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SUBSIDIZED SECURITY AND STABILITY OF EQUILIBRIUM SOLUTIONS IN AN *N*-PLAYER GAME WITH ERRORS

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Abstract

Optimizing resource allocation in interdependent security problems is a serious challenge for U.S. homeland security. In this paper, game theory is applied to this challenge in the case where investment by one defender has positive externalities for other players. The phenomena of tipping and cascading are discussed, and we explore how to target subsidized security in order to achieve the best results from tipping. We show that subsidization of security investments can increase the stability of equilibrium solutions, and also decrease the total expected social costs. The above findings are illustrated in numerical examples.

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Subsidized Security and Stability of Equilibrium Solutions in an N-Player Game with Errors

Jun Zhuang and Vicki M. Bier*

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1 Introduction

After September 11, 2001, homeland security has received a great deal of attention in the U.S. Since security-related resources are always limited, maximizing security subject to limited resources is a key challenge. "Tipping" (Schelling, 1978; Gladwell, 2002) has been suggested as a cost-effective way to encourage security investment, since if successful, a subsidy or other incentive to encourage a relatively small number of agents to invest can induce other agents to also invest in security, which may be beneficial when investing is the social optimum.

Many security problems (including aviation security, computer security, and supply-chain security) involve interdependence among potential defenders, meaning that one agent's strategy can affect the security environment for other agents. For example, poor security on the part of one airline, computer user, or supply-chain partner can increase the risk to other agents. Game theory has already been applied to such interdependent security problems (Kunreuther and Heal, 2003; Bier and Gupta, 2005). In particular, Kunreuther and Heal have noted that methods of coordination (such as regulation) are sometimes needed to help ensure that the optimum is achieved. However, it may be virtually impossible to enforce investment in security in some situations (especially when the number of agents is extremely large, as in computer security of numerous computer users), in which case subsidizing or incentivizing non-investing agents in some fashion to encourage them to invest may be a reasonable alternative.

One way of incentivizing investment is for security to be "bundled" with other goods and services (e.g., provided by one's internet service provider). Another way in which non-investing agents might be given an incentive to invest in security would be if such investment were a requirement for lucrative federal contracts, or for preferred-supplier relationships with industrial clients. (By analogy, federally funded academic researchers are often encouraged to engage in socially desirable practices such as educational outreach to under-represented minorities—either as a requirement of funding, or to increase the ratings of their proposals.)

The analysis in this paper shows that providing such incentives to a limited number of firms (e.g., government contractors) can make security investment sufficiently widespread that it becomes the norm even for firms that are not subject to such incentives. Thus, while incentives for investment are modeled in this paper as outright subsidies, we believe that the results apply much more broadly to other types of incentives besides direct subsidies (e.g., bundling of security with other services, or making security investment a requirement to be competitive for certain contracts).

To our knowledge, no previous studies have investigated the effect of subsidizing security investment on the stability and total social cost of equilibrium solutions. Previous studies have also not investigated the effect of error-prone agents on the stability of equilibrium solutions, nor how subsidization of security can be used to minimize the adverse effects of such errors. Finally, previous studies have not investigated how the total social costs in the case of subsidized security investment compare to the total costs in the absence of subsidization.

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The next section of this paper formulates a general interdependent security model for an arbitrary number of agents with attacks occurring over time. Section 3 solves this model and gives the strictly dominant strategies and equilibrium solutions, focusing on the general case of agents with heterogeneous time preferences. Section 4 discusses the phenomena of tipping and cascading in this context, and determines the minimal number of agents that must receive subsidized security in order to cause tipping. We also discuss which agents should be targeted in order to maximize the beneficial effects of tipping and cascading.

For simplicity, we then focus on the special case of homogeneous agents without a strictly dominant strategy. In this context, Section 5 studies the stability of equilibrium solutions, and explores whether subsidizing security investment can increase the stability of the socially optimum solution. Section 6 studies the effects of erroneous choices on the equilibrium solutions, and whether subsidizing security investment can reduce the adverse effects of erroneous choices. Section 7 studies the relationship between subsidization of security investment and the total expected social cost. Finally, Section 8 summarizes the previous sections and discusses the policy implications of our work. Proofs of all theorems are available in the appendix.

2 Assumptions, Notation and Model Formulation

2.1 Assumptions

Our basic model allows both direct attacks on an agent, and also indirect attacks (e.g., contamination from another agent in the system), as illustrated in Fig. 1. As in Bier and Gupta (2005), we assume that the time, t, of a direct attack on any agent follows an exponential distribution, and that direct attacks can lead to indirect attacks with some probability.

Of course, in the real world, terrorist attacks do not occur randomly, but as the result of careful planning by the attacker. However, considering attacks to be random with a constant rate of occurrence may not be unrealistic for some types of serious security threats, such as computer viruses. Therefore, for simplicity, our model views security solely as a game between defenders deciding whether to invest, and treats attacker behavior as exogenous. While this is obviously a somewhat limiting assumption, we believe that our model will be a valuable building block in generating insight into the relationships among defender choices, and in developing more complex models including endogenous attacker behavior as well as interactions among defenders. Models that treat security as a game between an intelligent attacker and a single defender include Bier et al. (2005), Bier et al. (2006), Konrad (2004), Woo (2002), Major (2002), Sandler and Arce M. (2003), Sandler and Lapan (1988), Lapan and Sandler (1993), and Zhuang and Bier (2006).

Like both Kunreuther and Heal (2003) and Bier and Gupta (2005), we assume that the loss is the same for both direct and indirect attacks, and that any attack is catastrophic (so that subsequent attacks affecting the same agent can be neglected). To justify the first assumption, we note, for example, that the loss from a computer virus is likely to be the same regardless of whether it is received directly from the virus developer, or inadvertently from another infected computer. Similarly, the loss from exposure to an infectious bioterrorist agent (such as smallpox or foot-and-mouth disease) may also be the same regardless of whether the exposure is the direct result of an intentional attack or the result of accidental contamination. To justify the assumption that even a single attack is catastrophic, we propose as examples situations in which a single successful attack might result in bankruptcy, death, loss of reputation, or theft of a valuable trade secret.

2.2 Notation and Model Formulation

We define the system parameters as follows:

- N: Number of agents in the system, where $N \geq 2$.
- h: Number of agents receiving subsidized security, where $0 \leq h \leq N$.
- M: Number of agents having security measures (either choosing to invest or receiving subsidized security from a third party), where $0 \leq M \leq N$.
- x: Number of agents making erroneous choices, where $0 \leq x \leq N - h$.
- λ_i: Rate of direct attacks on agent i, for $i = 1, ..., N$.

- $\tilde{\lambda}_i$: Total rate of all attacks on agent i (including indirect attacks), for $i = 1, ..., N$.
- q_{ij}: Probability that an attack on agent i infects agent j (where we define $q_{ii} = 1$), for $i, j = 1, ..., N$.
- r_i: Discount rate of agent i, where $r_i \geq 0$, for $i = 1, ..., N$.
- L_i: Loss suffered by agent i if it is attacked, either directly or indirectly, for $i = 1, ..., N$.
- C_i: Cost of investing in security for agent i. This investment is assumed to eliminate the risk of direct attacks, but have no effect on the risk of infection by indirect attacks from other agents. We assume that $0 < C_i < L_i \ \forall i = 1, ..., N$.
- s_i: Investment strategy for agent i, for $i = 1, ..., N$, where $s_i = 1$ if agent i invests in security, and $s_i = 0$ otherwise.
- $s_{-i} \equiv \{s_j, j \neq i\}$: Set of strategies of all agents other than agent i.
- $P_i(s_i, s_{-i})$: Total expected cost borne by agent i, for $i = 1, ..., N$, when it chooses strategy s_i (including both the cost of investment, if any, and the expected loss due to attacks), given the strategies of the other agents.

The expected loss experienced by agent i due to attacks is given by $L_i \int_0^\infty f_i(t) \exp(-r_i t) dt$, where $f_i(t) = \tilde{\lambda}_i \exp(-\tilde{\lambda}_i t)$ is the probability density function for the time of the first attack on agent i, and $\tilde{\lambda}_i = (1 - s_i)\lambda_i + \sum\limits_{j \neq i} (1 - s_j) q_{ji} \lambda_j$ is the total rate of attacks against agent i. Hence, the net present value of the expected loss due to attacks experienced by agent i is

$$E(Loss) = L_i \int_0^\infty \tilde{\lambda}_i \exp(-\tilde{\lambda}_i t - r_i t) dt = L_i / (1 + r_i / \tilde{\lambda}_i) \tag{1}$$

and the total expected cost to agent i is given by

$$P_i(s_i, s_{-i}) = s_i C_i + L_i / (1 + r_i / \tilde{\lambda}_i) \tag{2}$$

3 Equilibrium Solutions

Definition 1. A *pure-strategy Nash equilibrium* (abbreviated as *equilibrium* in our paper) is a set of strategies $\{s_i, i = 1, ..., N\}$ such that no one agent would be better off by switching strategies unless at least one other agent also switched. Thus, at equilibrium, we must satisfy the following system of inequalities:

$$P_i(s_i, s_{-i}) \leq P_i(1 - s_i, s_{-i}) \ \forall i = 1, ..., N \tag{3}$$

Definition 2. Strategy s_i is a *strictly dominant* strategy for agent i if and only if $P_i(s_i, s_{-i}) < P_i(1 - s_i, s_{-i})$ for all s_{-i}.

For simplicity, in the remainder of this paper, we consider the case where only the discount rates r_i of the agents differ. In other words, we let $\lambda_i = \lambda > 0$, $C_i = C > 0$, $L_i = L > 0$, and $q_{ij} = q > 0$ for $i, j = 1, ..., N$, $i \neq j$, since the effects of these parameters have already been extensively investigated by Kunreuther and Heal (2003) and Bier and Gupta (2005). Thus, equation (2) becomes:

$$P_i(s_i, s_{-i}) = s_i C + L / (1 + r_i / \tilde{\lambda}) \tag{4}$$

where $\tilde{\lambda} = (1 - s_i)\lambda + \sum\limits_{j \neq i} (1 - s_j) q \lambda$.

Suppose that in some equilibrium solution, exactly M agents receive subsidized security or choose to invest (for some $0 \leq M \leq N$). Then there are three possible cases:

1. All agents have security measures in place ($M = N$);
2. Some agents have security measures and some not ($1 \leq M \leq N - 1$); or

3. No agents have security measures in place ($M = 0$).

For convenience, we renumber the agents so that the first M agents have security measures, and the remaining $N - M$ agents do not (i.e., $s_i = 1 \; \forall i = 1, ..., M$, and $s_i = 0 \; \forall i = M+1, ..., N$). We begin by considering a model that does not include the possibility of subsidized security (i.e., with $h = 0$). Table 1 gives the costs to the agents in all three of the above cases for that model. It also specifies the cost to any agent i that deviates from the equilibrium action s_i, conditional on the other agents' strategies; i.e., $P_i(1 - s_i, s_{-i})$.

Solving the system of inequalities (3) for $i = 1, ..., N$ using the costs in Table 1 yields conditions for the discount rates at which agents will be willing to invest in security at equilibrium. These conditions are summarized in Table 2, using the following notation:

- $\tilde{N} \equiv \lfloor C(L/C - 1)^2/(4Lq) \rfloor$ (where $\lfloor x \rfloor$ is the greatest integer less than or equal to x);
- $R_1(k) \equiv \lambda \left[L/C - 1 - 2kq - \sqrt{(L/C-1)^2 - 4qkL/C} \right] /2$ for $k = 0, ..., \tilde{N}$; and
- $R_2(k) \equiv \lambda \left[L/C - 1 - 2kq + \sqrt{(L/C-1)^2 - 4qkL/C} \right] /2$ for $k = 0, ..., \tilde{N}$.

Here, $\tilde{N}$ is a bound on the number of agents there can be in a system for certain properties to hold, and $R_1(k)$ and $R_2(k)$ are the minimum and maximum discount rates for which an agent would invest given that exactly others are not doing so, respectively. By taking derivatives, it is easy to show that $R_1(k)$ and $R_2(k)$ are increasing and decreasing in k, respectively. Also, note that $R_1(\tilde{N}) \leq R_2(\tilde{N})$, and $R_1(0) = 0$. Thus, the following relationship holds, as shown in Fig. 2:

$0 = R_1(0) < R_1(0) < ... < R_1(\tilde{N}) \leq R_2(\tilde{N}) < ... < R_2(1) < R_2(0)$

We define the following sets on the domain of discount rates:

- $S_l = (R_2(0), \infty)$ for $l = 0$
- $S_l = (R_1(l-1), R_1(l)) \cup (R_2(l), R_2(l-1))$ for $l = 1, ..., \min(N-1, \tilde{N})$, and
- $S_l = (R_1(l-1), R_2(l-1))$ for $l = \min(N, \tilde{N}+1)$

Intuitively, S_l is the set of those discount rates for which an agent would want to invest if at most $l - 1$ others do not invest, but would not want to invest if l or more others do not invest. This notation is represented graphically in Fig. 2 for the case where $N \leq \tilde{N} + 1$. When we have $N \geq \tilde{N} + 2$, then $R_1(k)$ and $R_2(k)$ are not defined for $\tilde{N} + 1 \leq k \leq N$, so S_l is empty for $\tilde{N} + 2 \leq l \leq N$. For convenience, we also define $\mathbf{Cl}(S_l)$ to be the closure of the open set S_l.

Definition 3. Given a system in which exactly M agents have security measures in place at equilibrium:

- Let $Inv(M)$ be the set of possible discount rates for the investing agents; [1]
- Let $Non(M)$ be the set of possible discount rates for the (non-subsidized) agents that do not invest; and
- Let $Cmpl(M) = [Inv(M) \cup Non(M)]^C$ be the set of discount rates that can not be held by any (non-subsidized) agent if exactly M agents have security measures at equilibrium.

Using the notation above, Table 2 can be rewritten as shown in Table 3, specifying the ranges of discount rates possible for both the investing and non-investing agents, given the number of agents M having security measures. These ranges are also illustrated in Figures 3 and 4, for the investing and non-investing agents, respectively. (As before, when $N \geq \tilde{N} + 1$, some of the S_l will be empty.) Note also that the existence of an equilibrium strategy with M investing agents implies that no (non-subsidized) agent has a discount rate in the set $Cmpl(M)$.

[1] Note that there may be fewer than M investing agents if subsidized agents are also considered, as in subsequent sections of this paper. However, those agents actually investing would still need to have discount rates in $Inv(M)$ if a total of M agents have security in place, regardless of whether some of those M agents obtained their security through subsidies.

Theorem 1. *If and only if $r_i \in S_N$ (respectively, $r_i \in S_0$), $s_i = 1$ (respectively, $s_i = 0$) is a strictly dominant strategy for agent i in any equilibrium, for $i = 1, ..., N$.*

Theorem 2. *Holding all else constant, as the number of agents N in the system increases, the range of discount rates S_N for which investing is a strictly dominant strategy becomes smaller. If $N \geq \tilde{N} + 2$, then there is no discount rate for which investing in security is a strictly dominant strategy. The range of discount rates S_0 for which not investing is a strictly dominant strategy does not depend on N.*

Remark: This theorem implies that for systems with sufficiently large numbers of agents, investing in security will not be a strictly dominant strategy for any agent. Agents can of course still choose to invest, but will do so in equilibrium only if other agents also have security in place. This finding helps to justify the need for coordinating mechanisms (such as subsidies) in large systems.

Example 1. We use the following parameter values to illustrate the results in Theorem 2: $C = 10$; $L = 1000$; $q = 0.01$; $\lambda = 0.01$; $N = 2000$; and therefore $\tilde{N} = 2450$. Fig. 5 shows the regions of discount rates in which investing and not investing, respectively, are dominant strategies, as functions of the number of agents.

The remainder of this paper focuses on the case where no non-subsidized agent has a strictly dominant strategy. In other words, we assume that the discount rate of agent i satisfies $r_i \notin S_0$ and $r_i \notin S_N$ $\forall i = h + 1, ..., N$.

Theorem 3. *In an N-agent model, if no agent has a strictly dominant strategy of not investing, then $\{$all invest$\}$ will be an equilibrium solution. The total cost borne by any given agent in the equilibrium $\{$all invest$\}$ will be lower than the corresponding cost in any other possible equilibrium solution, with the difference in cost growing in N.*

Theorem 4. *In a homogeneous N-agent model (i.e., $r_i = r$ $\forall i = 1, ..., N$), if the agents do not have a strictly dominant strategy, then there exist exactly two equilibrium solutions: $\{$all invest$\}$; and $\{$none invest$\}$.*

4 Tipping and Cascading

We now discuss the possibility of tipping, and its effect on the equilibrium solution. In general, starting with an equilibrium in which M agents invest (for $M \leq N - 1$), ensuring that some additional agents invest will tend to make investing more attractive for the remaining agents. In practice, there may be different ways of ensuring that agents invest, such as mandating investment in security, or providing subsidized security. In this paper, we focus on the latter case, and assume in particular that security becomes free to some agents.

Definition 4. Let $Inv_c(M, h)$ be the set of possible discount rates for those agents who would not invest at equilibrium with $h = 0$, but would find investing attractive if the number of other agents having security measures in place increased from M to $M + h$.

Note that $Inv_c(M, h)$ will include those with discount rates in $Cmpl(M + h)$, since there can be an equilibrium strategy with at least $M + h$ agents having security measures only if no other non-subsidized agents have discount rates in the set $Cmpl(M + h)$. Thus, we will have:

$$Inv_c(M, h) = [Inv(M + h) - Inv(M)] \cup Cmpl(M + h) \tag{5}$$

Here, "$-$" is the set operator representing the difference between sets $Inv(M + h)$ and $Inv(M)$. If we let $\Theta(S)$ denote the number of agents with discount rates in the set S, then $\Theta\left[Inv_c(M, h)\right]$ is the number of agents that could be induced to invest by tipping.

Remark: It is straightforward to show that $\Theta\left[Inv_c(M, h)\right]$ is non-decreasing in h for any given value of M, with $\Theta\left[Inv_c(M, 0)\right] = 0$.

Starting with an equilibrium in which exactly M agents invest, the number of agents h that must receive subsidized security in order to lead to tipping must satisfy $\Theta\left[Inv_c(M, h)\right] > 0$. Moreover, if $\Theta\Big\{Inv_c\big[M + h, \Theta[Inv_c(M, h)]\big]\Big\} > 0$, then additional agents will choose to invest, in which case the number of agents having security measures will increase from $M + h$ to $M + h + \Theta\left[Inv_c(M, h)\right]$, and so on. We call this phenomenon "cascading."

Note that no tipping or cascading will occur if $\Theta\left[Inv_c(M,h)\right]=0$. The minimal number of agents that must receive free security in order to lead to tipping is given by $\min\Big\{h : \Theta\big[\big(Inv_c(M,h)\big)\big] > 0\Big\}$. The discount rates of the subsidized agents are irrelevant to determining whether tipping occurs. However, the discount rates of the subsidized agents do determine whether cascading occurs, and how far it progresses. Therefore, it makes sense if possible to target any subsidies at those agents who are least likely to begin investing due to tipping (i.e., agents with discount rates in or near the region where not investing is the strictly dominant strategy). These arguments are illustrated below; see also Kunreuther and Heal (2003) and Dixit (2003).

Example 2. Consider an N-agent system with $h = 0$, $N \leq \tilde{N}+1$ and $r_i \in S_{i-1}$ $\forall i = 1, ..., N$. Perusal of Fig. 3 indicates that for this example, there exists no equilibrium with a non-zero number of agents investing. Furthermore, Fig. 4 shows that the solution {none invest} (with $M = 0$ agents investing) is an equilibrium for this example. Now suppose that agent 1 receives free security. Then agent N will also choose to invest, because $r_N \in S_{N-1} \subset Inv_c(0,1)$. In other words, since agent 1 is receiving subsidized security, not investing is no longer optimal for agent N (from Fig. 4). Since agent N is better off investing, there will now be $M = 2$ agents having security measures. Therefore, agent $N-1$ will also begin investing, because $r_{N-1} \in S_{N-2} \subset Inv_c(1, \Theta\left[Inv_c(0,1)\right])$. Similarly, agent $N-2$ will begin investing once there are $M = 3$ agents having security measures, and so on. Thus, if agent 1 receives free security, {all invest} becomes the unique equilibrium in this example. Note also that the discount rate of the subsidized agent determines how many agents will decide to invest as a result of cascading. In this example, it is straightforward to see that if agent i is the one that receives free security, the system will end up in a unique equilibrium with $M = N + 1 - i$ agents having security measures.

5 Stability of Equilibrium Solutions

For simplicity, we now consider the case of homogeneous time preferences in an N-agent model in which no agent has a strictly dominant strategy. In other words, we let $r_i = r \in \mathbf{Cl}(S_k)$ $\forall i = 1, ..., N$ for $1 \leq k \leq \min(N-1, \tilde{N})$.

Definition 5. In an N-agent homogeneous model, let n be the greatest integer such that, even if n agents all change to the opposite strategy, the remaining $N - n$ agents will not want to change their strategies. We then define the stability level of an equilibrium (either {all invest} or {none invest}) to be $\alpha = n/(N-1)$.[2]

Remark: If $\alpha = 0$, then the corresponding equilibrium is completely unstable; that is, if even one agent changes strategy, then at least one other agent will also prefer to change strategy. If $\alpha = 1$, the corresponding equilibrium is completely stable; that is, no matter how many agents change strategies, no other (rational) agent would want to change strategy. Note that the stability of Nash equilibrium solutions has been defined variously in other research; see for example Damme (1991), Kohlberg and Mertens (1986), and Okada (1981).

Since the h agents receiving subsidized (free) security need not incur any cost to obtain security, we assume that all subsidized agents will have security measures in place, but will incur non-investment cost. We now consider the strategies of the $N-h$ non-subsidized agents. Moreover, we use the notation $\{\text{all invest}\}_{N-h}$ and $\{\text{none invest}\}_{N-h}$ to denote sub-equilibrium solutions describing the possible behavior of these $N-h$ agents.

Theorem 5. *Consider a model with N homogeneous agents, where $r_i = r \in \mathbf{Cl}(S_k)$ $\forall i = 1, ..., N$ for $1 \leq k \leq \min(N-1, \tilde{N})$. Then $\{\text{all invest}\}_{N-h}$ will be a sub-equilibrium for the $N-h$ non-subsidized agents for any value of h. This sub-equilibrium has stability $\alpha = \frac{k-1}{N-h-1}$ if $h \leq N-k-1$, and $\alpha = 1$ if $h \geq N-k$. By contrast, $\{\text{none invest}\}_{N-h}$ is a sub-equilibrium only if $h \leq N-k-1$, in which case its stability is given by $\alpha = \frac{N-h-k-1}{N-h-1}$.*

Remark: If $r_i = r \in \mathbf{Cl}(S_k)$ $\forall i = 1, ..., N$, then the stability of $\{\text{all invest}\}_{N-h}$ is increasing in h for $h \leq N-k-1$, and equals 1 (complete stability) when $h \geq N-k$ agents receive subsidized (free) security.

[2] Our definition of the stability α is closely related to what game theorists call "p-dominance" (see Morris et al., 1995).

Theorem 6. *If both {all invest}$_{N-h}$ and {none invest}$_{N-h}$ are possible sub-equilibrium solutions, then {all invest}$_{N-h}$ will be more stable[3] than {none invest}$_{N-h}$ when $k > \frac{N-h}{2}$. Conversely, {none invest}$_{N-h}$ will be more stable than {all invest}$_{N-h}$ when $k < \frac{N-h}{2}$. If $N-h$ is even, then the two sub-equilibrium solutions will be equally stable when $k = \frac{N-h}{2}$.* (Proof follows directly from Theorem 5.)

Remark: Generally, {all invest}$_{N-h}$ will tend to be more stable than {none invest}$_{N-h}$ when the discount rate of the (homogeneous) agents is close to the region where investing is strictly dominant. Similarly, {none invest}$_{N-h}$ will tend to be more stable than {all invest}$_{N-h}$ when the discount rate is close to the region where not investing is strictly dominant. If $N-h$ is even, then there exists a middle range, $S_{(N-h)/2}$, where the two sub-equilibrium solutions are equally stable. This is illustrated in Fig. 6 for the case $h = 0$.

6 Erroneous Choice

In a model with N homogeneous agents, if $r \in \mathbf{Cl}(S_k)$ for $1 \leq k \leq \min(N-1, \tilde{N})$, then Theorem 3 shows that the equilibrium {all invest} is the social optimum, and moreover has lower cost than the equilibrium {none invest} for any given agent individually. Therefore, it may be reasonable to expect that any rational agent would choose to invest in this case. However, in practice, some agents may choose not to invest even when it would be in their interests to do so. We denote such behavior an *erroneous choice*. In this section, rather than assuming that all of the N (homogenous) agents make the same choice (as in Section 5), we assume that some agents (at random) erroneously choose not to invest, and the remaining agents choose strategy the sub-equilibrium with the lowest social cost in light of the observed number of erroneous choices. We here examine the effect of erroneous choice on the sub-equilibrium solutions for the remaining agents who do not make errors. We also examine how subsidization of security investment can help to counteract any adverse effect of erroneous choices.

Since the h agents receiving subsidized (free) security need not incur any cost to obtain security, we consider only the strategies of the $N-h-x$ non-subsidized agents who do not make erroneous choices. Let {all invest}$_{N-h-x}$ and {none invest}$_{N-h-x}$ be sub-equilibrium solutions describing the possible behavior of the $N-h-x$ non-subsidized agents not making erroneous choices.

Theorem 7. *Both {all invest}$_{N-h-x}$ and {none invest}$_{N-h-x}$ will be sub-equilibrium solutions if and only if $x \leq k-1$ and $h \leq N-k-1$, respectively. Therefore, there will always exist at least one sub-equilibrium solution. Moreover, if $x+1 \leq k \leq N-h-1$, then {all invest}$_{N-h-x}$ and {none invest}$_{N-h-x}$ are both possible sub-equilibrium solutions. In this case, the total cost borne by any of the N agents individually in {all invest}$_{N-h-x}$ is lower than the corresponding cost when the $N-h-x$ agents do not invest. Thus, {all invest}$_{N-h-x}$ is the socially optimal sub-equilibrium.*

Remark: If those non-subsidized agents not making erroneous choices always choose the social optimum, then they will choose to invest whenever {all invest}$_{N-h-x}$ is a sub-equilibrium.

Theorem 8. *Suppose that each non-subsidized agent independently makes an erroneous choice with probability ε, where $0 \leq \varepsilon \leq 1$. In this case, the number of agents X making erroneous choices is a random variable with binomial probability mass function given by $P(X = x) = \binom{N-h}{x}\varepsilon^x(1-\varepsilon)^{N-h-x}$. Let P_{Inv} be the probability that {all invest}$_{N-h-X}$ is a sub-equilibrium for those non-subsidized agents not making erroneous choices. Then, we have $P_{Inv} = 1$ if $h \geq N-k$, and $P_{Inv} = \sum_{x=0}^{k-1} \binom{N-h}{x}\varepsilon^x(1-\varepsilon)^{N-h-x}$ if $h \leq N-k-1$.*

Remarks: If fewer than $N-k$ agents receive subsidized (free) security, then P_{Inv} will be increasing in the number of agents h receiving free security (all else constant), in part because provision of free security to a larger number of agents reduces the maximum possible number of agents who could make erroneous choices. P_{Inv} is also increasing in k, where $r \in \mathbf{Cl}(S_k)$ is the discount rate of the (homogeneous) agents; that is, as r gets closer to the region where investing is dominant (all else constant), it becomes more likely that investing will be a sub-equilibrium for the $N-h-X$ non-subsidized agents not making erroneous choices. All else constant, P_{Inv} is also decreasing in both the error probability ε and the number of agents N. The above observations are based on known properties of the binomial distribution (Bickel and Doksum, 2001).

[3] Also sometimes called "risk dominant" (see Harsanyi and R.Selten, 1988).

7 Total Social Cost

In this section, we explore the effect of providing subsidized (free) security to a subset of agents on the total (expected) social cost. Let $C_F(h)$ (with $C_F(h) - C_F(h-1) \leq C$, for $h \geq 1$) be the cost to a third party of providing subsidized (free) security to h agents, $C_x(h)$ be the cost to the x agents who make erroneous choices, $C_O(h)$ be the cost to the other $N - h - x$ agents, and $C_S(h) \equiv C_F(h) + C_x(h) + C_O(h)$ be the total (expected) social cost. We consider three possible functional forms for $C_F(h)$, all of which satisfy $C_F(h) \leq hC$:

1. $C_F(h) = hC$; i.e., the cost to a third party of providing security is the same as the cost to the agents themselves.
2. $C_F(h)$ is increasing and concave; i.e., a third party can provide security at lower cost than the individual agents could (e.g., due to economies of scale).
3. $C_F(h)$ is constant in h (in this case, we will have $C_F(h) \leq hC$ for h sufficiently large); i.e., provision of security by a third party is cost-effective only for systems with a large number of agents. This is a bounding case, in which once security technology (e.g., anti-virus software) has been developed, it can be provided to any number of agents at no additional cost. In this case, it would be clearly optimal to give free security to all agents if the number of agents is sufficiently large.

For the sub-equilibrium solution $\{\text{all invest}\}_{N-h-x}$ (if it exists), the number of agents having security measures will be $M = N - x$. Then using Table 1 (but subtracting the investment cost C for those h agents that receive subsidized security), we will have $C_x(h) = xL/(1 + r/\tilde{\lambda})$ for $\tilde{\lambda} = \lambda + (x-1)q\lambda$, and $C_O(h) = (N - h - x)[C + L/(1 + r/\tilde{\lambda})]$ for $\tilde{\lambda} = xq\lambda$. Similarly, for the sub-equilibrium solution $\{\text{none invest}\}_{N-h-x}$ (if it exists), the number of agents having security measures in place will be given by $M = h$. Then using Table 1 (but again subtracting the investment cost C for the subsidized agents), we will have $C_x(h) = xL/(1+r/\tilde{\lambda})$ for $\tilde{\lambda} = \lambda+(N-h-1)q\lambda$, and $C_O(h) = (N-h-x)L/(1+r/\tilde{\lambda})$ for $\tilde{\lambda} = \lambda+(N-h-1)q\lambda$.

Theorem 9. *In both sub-equilibrium solutions $\{\text{all invest}\}_{N-h-x}$ and $\{\text{none invest}\}_{N-h-x}$ (if they exist), the total social cost $C_S(h)$ is non-increasing in the number h of agents receiving subsidized (free) security if $C_F(h)$ satisfies $C_F(h) - C_F(h-1) \leq C$, for $h \geq 1$.*

Remark: From Theorems 7 and 9, we know that subsidization of security will decrease the total social cost $C_S(h)$ in both $\{\text{all invest}\}_{N-h-x}$ and $\{\text{none invest}\}_{N-h-x}$, and moreover will ensure that $\{\text{all invest}\}_{N-h-x}$ is the unique sub-equilibrium whenever $h \geq N - k$. This suggests that in order to minimize the total (expected) social costs, all agents should receive subsidized (free) security (i.e., $h = N$) whenever investing in security is the social optimum, as long as security can be provided at lower cost by a third party than by the agents themselves.

Example 3. In this example, we numerically explore the effects of offering subsidized (free) security in the case discussed in Section 5 (where all non-subsidized agents make the same choice), using parameter values: $C = 10$, $L = 1000$, $q = 0.01$, $\lambda = 0.01$, $k = 1200$, and $N = 2000$. Fig. 7 shows the stability of the sub-equilibrium solutions $\{\text{all invest}\}_{N-h}$ and $\{\text{none invest}\}_{N-h}$ as a function of h. Providing subsidized (free) security increases the stability of $\{\text{all invest}\}_{N-h}$, and decreases the stability of $\{\text{none invest}\}_{N-h}$, as shown in Theorem 5. Fig. 8 shows the cost $C_A(h)$ actually borne by the agents as a function of h. Fig. 8 also shows the total social cost $C_S(h)$ as a function of h in $\{\text{all invest}\}_{N-h}$ for three different assumptions about $C_F(h)$; note that $C_S(h)$ is non-increasing in all three cases. In addition, for the sub-equilibrium $\{\text{none invest}\}_{N-h}$, $C_S(h)$ is decreasing in all three cases; however, the results are not shown in Fig. 8, since for $\{\text{none invest}\}_{N-h}$, $C_S(h)$ is approximately equal to $C_A(h)$ in all three cases, so would not be clearly visible in the figure.

8 Conclusions

Sections 2-4 of this paper formulate and solve an interdependent security model for an arbitrary number of agents with attacks occurring over time, focusing on the case of agents with heterogeneous time preferences. Results show that when multiple equilibrium solutions exist, the social optimum is for all agents to invest, as long as that is an equilibrium solution. The role of tipping and cascading in helping to achieve this social optimum is discussed. In particular, we explore the minimal number of agents who would need to receive subsidized (free) security in order for tipping to occur, and which agents should ideally receive such free security. (However, we recognize that the mechanisms by which subsidies could be implemented in the real world may not allow such careful targeting.) In this paper, we focus primarily on the effect of discount rates; in particular, we show that the mere existence of agents with extreme discount rates (e.g., due to myopia) can make it undesirable for other agents to invest. However, similar results also hold for heterogeneity in other parameters (such as the cost of investing in security, or the loss resulting from a successful attack).

Sections 5-7 further investigate the effect of providing subsidized (free) security on both the stability of equilibrium solutions and the total social cost, in the case of homogeneous discount rates. Results show that subsidization can increase the stability of the socially optimum equilibrium solution (in which all non-subsidized agents invest), reduce or eliminate the adverse effects of erroneous choices, and decrease the total (expected) social cost of achieving the social optimum. Our work suggests that under appropriate circumstances, providing subsidized security to some agents can: (1) ensure that even agents for which not investing would otherwise be strictly dominant do actually invest (through careful targeting of the subsidies to those agents); (2) lead to tipping and cascading, thereby causing additional agents to invest; (3) increase the stability of the socially optimum equilibrium in which all agents invest; (4) counteract the effects of erroneous choices; and (5) decrease the (expected) total social cost. Thus, it might sometimes be worthwhile for third parties (such as governments) to subsidize the provision of security, or otherwise ensure that the strategy of investing in security is adopted when it is the social optimum, since that might not otherwise occur, especially in systems with large numbers of agents. As an example, one mechanism for implementing this idea in practice might be to require recipients of federal grants and contracts to adopt socially desirable security practices.

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Table 1: Costs for agents when M agents having security

	For investing agents $(i = 1, ..., M)$, $s_i = 1$		For not-investing agents $(i = M+1, ..., N)$, $s_i = 0$	
	$P_i(s_i, s_{-i})*$	$P_i(1-s_i, s_{-i})$	$P_i(s_i, s_{-i})$	$P_i(1-s_i, s_{-i})$
Case (1): $M = N$	C	$L/(1+r_i/\tilde{\lambda})$ $\tilde{\lambda} = \lambda$	N/A	N/A
Case (2): $1 \leq M \leq N-1$	$C + L/(1+r_i/\tilde{\lambda})$ $\tilde{\lambda} = (N-M)q\lambda$	$L/(1+r_i/\tilde{\lambda})$ $\tilde{\lambda} = \lambda + (N-M)q\lambda$	$L/(1+r_i/\tilde{\lambda})$ $\tilde{\lambda} = \lambda + (N-M-1)q\lambda$	$C + L/(1+r_i/\tilde{\lambda})$ $\tilde{\lambda} = (N-M-1)q\lambda$
Case (3): $M = 0$	N/A	N/A	$L/(1+r_i/\tilde{\lambda})$ $\tilde{\lambda} = \lambda + (N-1)q\lambda$	$C + L/(1+r_i/\tilde{\lambda})$ $\tilde{\lambda} = (N-1)q\lambda$

*The costs for subsidized agents are the same with $P_i(s_i, s_{-i})$ of investing agents subtracting the investment cost C.

Table 2: Conditions for an equilibrium in which M agents have security

	For investing agents $(i = 1, ..., M)$, $s_i = 1$	For not-investing agents $(i = M+1, ..., N)$, $s_i = 0$
Case (1): $M = N$	$R_1(0) \leq r_i \leq R_2(0)$	N/A
Case (2):* $1 + (N - \tilde{N})^+ \leq M \leq N-1$	$R_1(N-M) \leq r_i \leq R_2(N-M)$	$r_i \leq R_1(N-M-1)$ or $r_i \geq R_2(N-M-1)$
Case (3): $M = 0$	N/A	$r_i \leq R_1(N-1)$ or $r_i \geq R_2(N-1)$

*No equilibrium is possible with $1 \leq M \leq (N - \tilde{N})^+$.

Table 3: Conditions for an equilibrium in which M agents have security, in terms of the $\{S_l\}$

	$Inv(M)$ (Ranges of discount rates of investing agents)	$Cmpl(M)$ (Ranges of discount rates not held by any agents)	$Non(M)$ (Ranges of discount rates of non-investing agents)
Case (1): $M = N$	$\bigcup_{l=1}^{\min\{\tilde{N}+1, N\}} \mathbf{Cl}(S_l)$	S_0	N/A
Case (2):* $1 + (N - \tilde{N})^+ \leq M \leq N-1$	$\bigcup_{l=N-M+1}^{\min\{\tilde{N}+1, N\}} \mathbf{Cl}(S_l)$	S_{N-M}	$\bigcup_{l=0}^{N-M-1} \mathbf{Cl}(S_l)$
Case (3): $M = 0$	N/A	S_N	$\bigcup_{l=0}^{\min\{\tilde{N}+1, N-1\}} \mathbf{Cl}(S_l)$

*No equilibrium is possible with $1 \leq M \leq (N - \tilde{N})^+$.

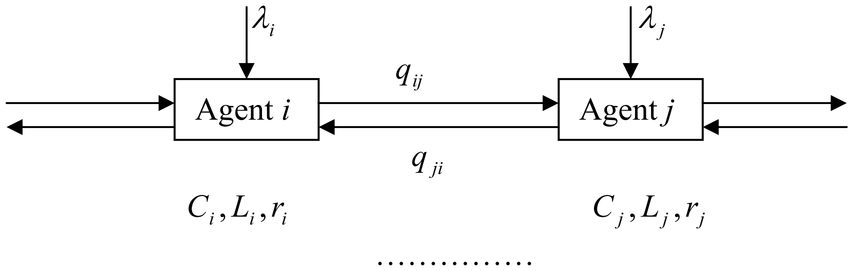

Figure 1: Model structure

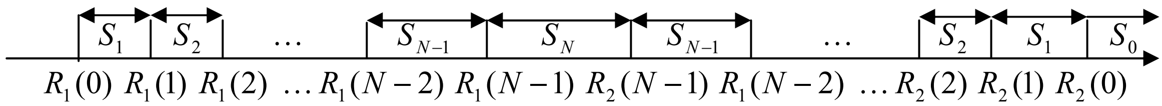

Figure 2: Illustration of the ranges S_l

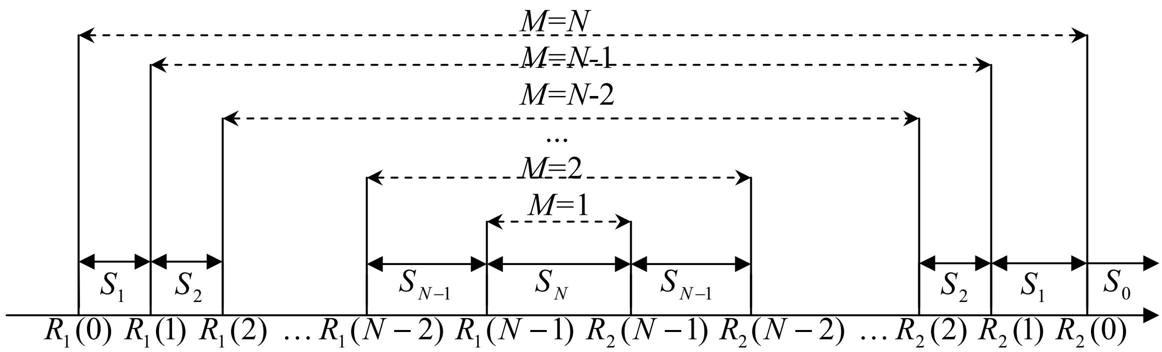

Figure 3: Ranges of discount rates possible for investing agents when M agents invest or receive subsidized security

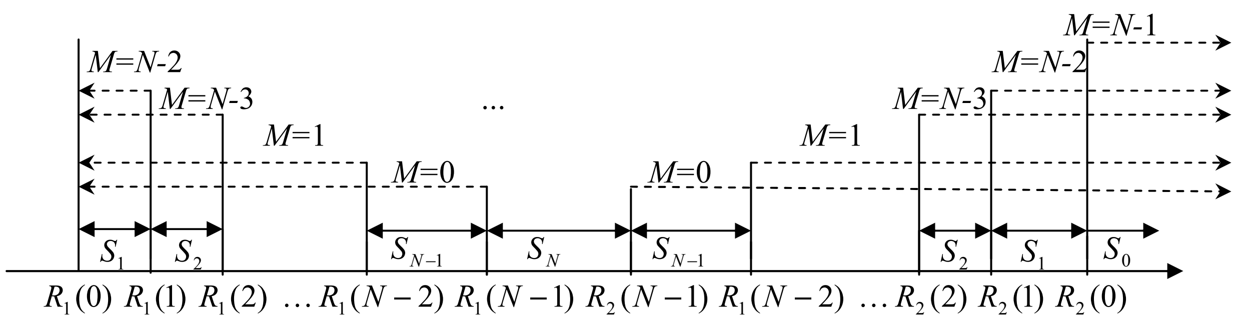

Figure 4: Ranges of discount rates possible for non-investing agents when M agents invest or receive subsidized security

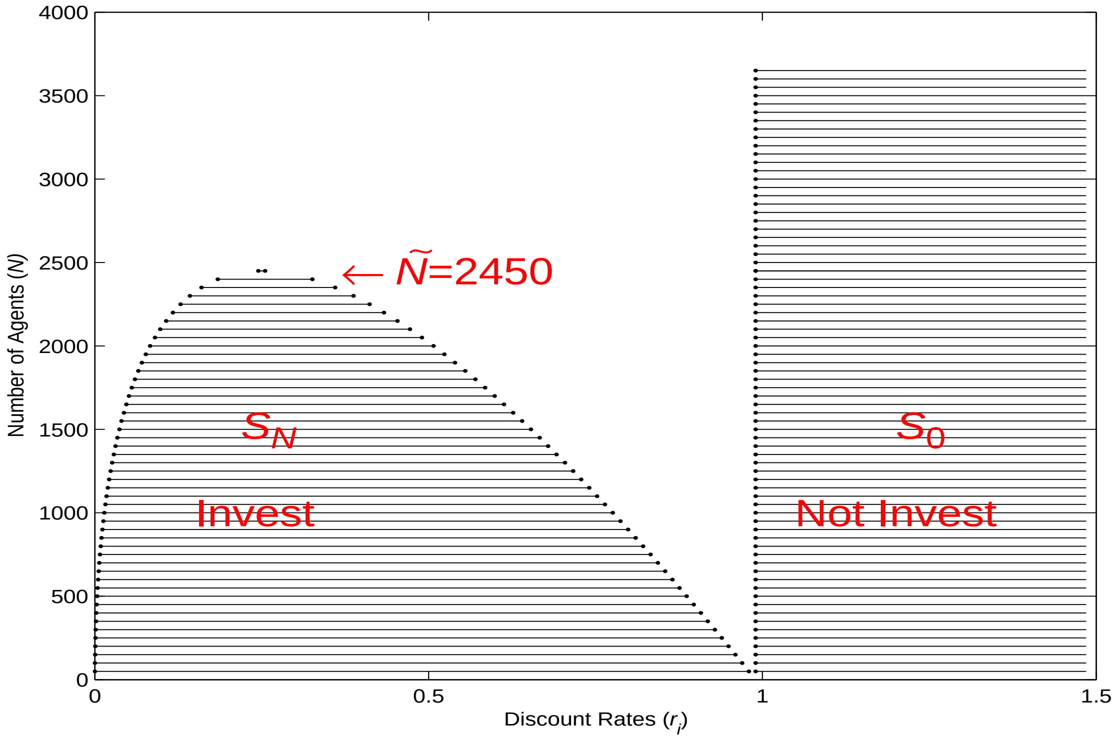

Figure 5: Discount rates for which investing and not investing are dominant, as a function of N

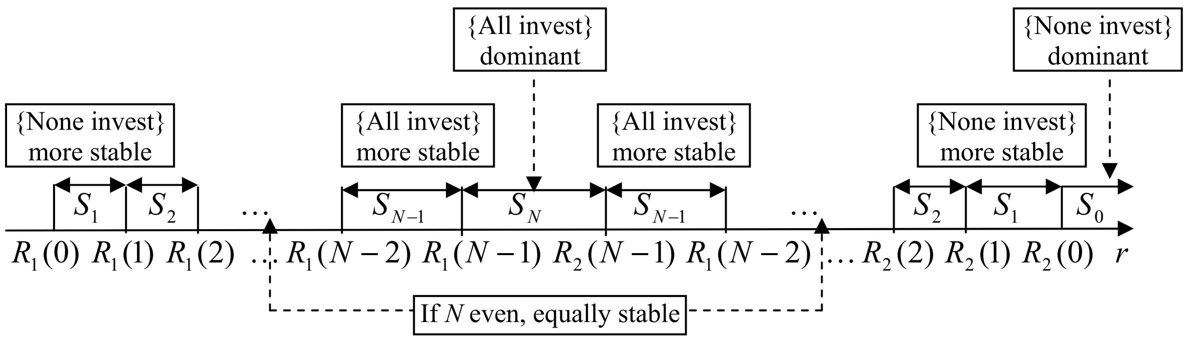

Figure 6: Stability of equilibrium solutions in an N-agent homogeneous model

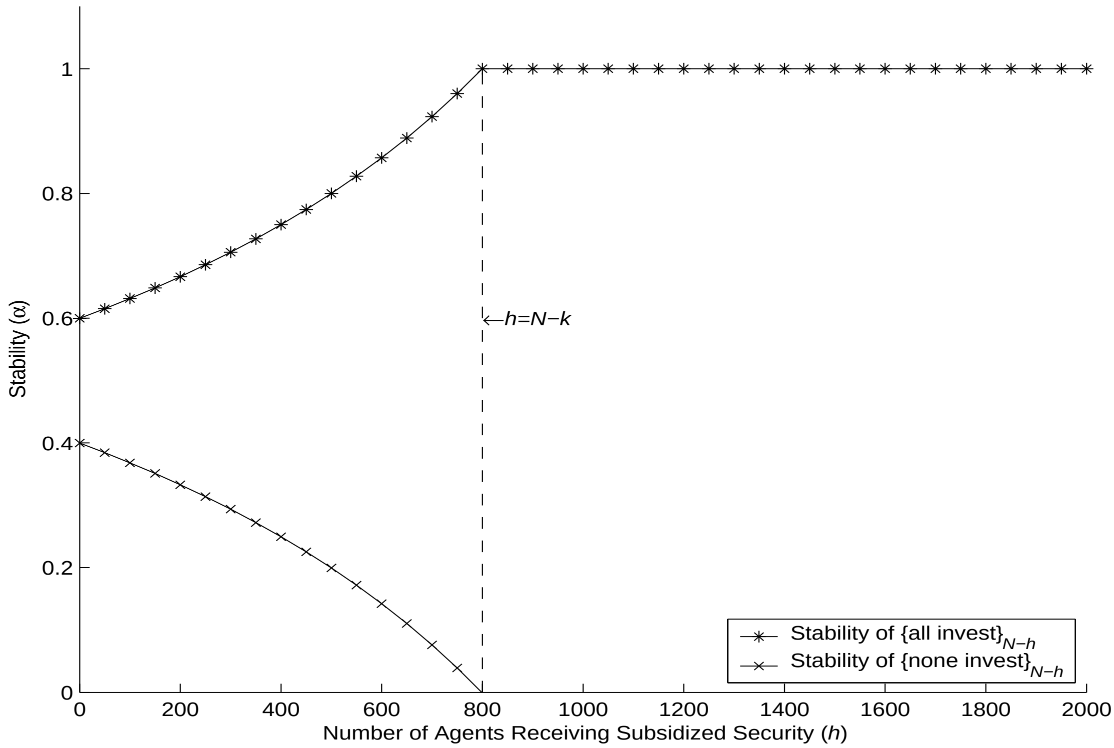

Figure 7: Stability of the sub-equilibrium solutions $\{\text{all invest}\}_{N-h}$ and $\{\text{none invest}\}_{N-h}$

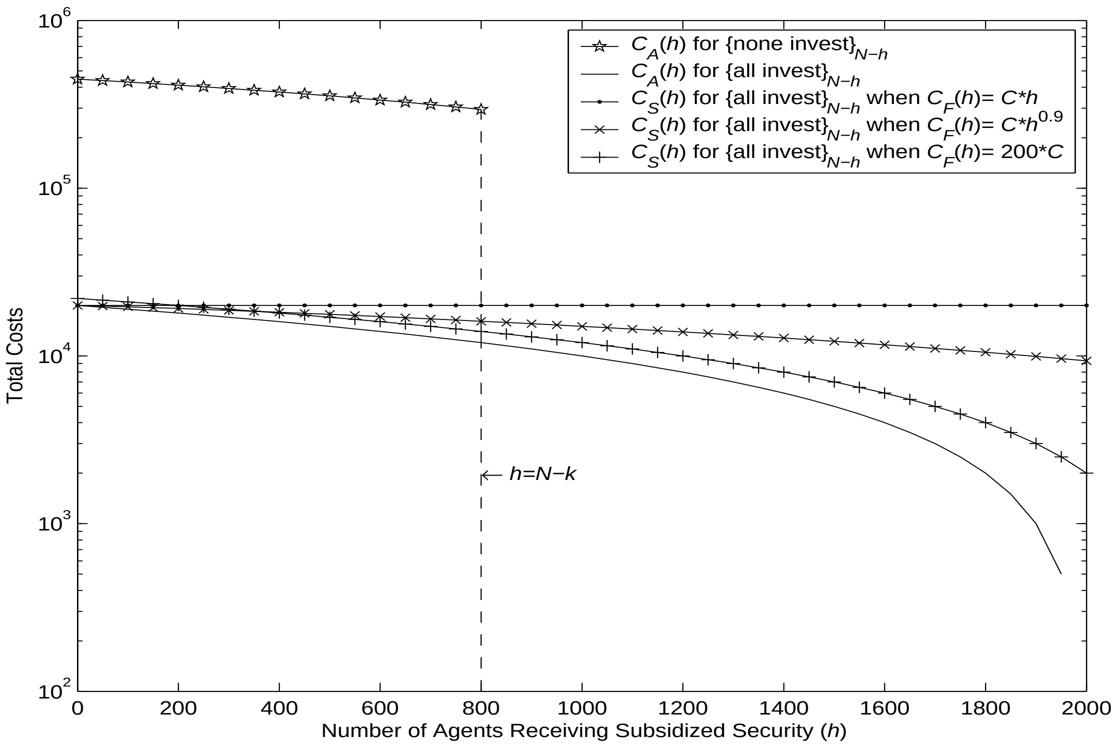

Figure 8: Total costs of sub-equilibrium solutions $\{\text{all invest}\}_{N-h}$ and $\{\text{none invest}\}_{N-h}$

9 Appendix

9.1 Proof of Theorem 1

According to Table 3, if $r_i \in S_N$ for some agent i, then not investing can never be a best response no matter how many of other agents choosing invest. Therefore, investing must be a strictly dominant strategy. Similar argument holds for the remaining parts of the theorem. $\square$

9.2 Proof of Theorem 2

Since $R_1(k)$ and $R_2(k)$ are increasing and decreasing in k for $k = 1, ..., \tilde{N}$, respectively, the set $S_N = (R_1(N-1), R_2(N-1))$ becomes smaller as N increases. For $N \geq \tilde{N} + 2$, the set S_N is empty. The set $S_0 = (R_2(0), +\infty)$ does not depend on N, by the definitions of $R_1(k)$ and $R_2(k)$. $\square$

9.3 Proof of Theorem 3

Theorem 1 indicates that not investing is a strictly dominant strategy for agent i if and only if $r_i \in S_0$. Since we assume that no agent has a strictly dominant strategy of not investing, we will have $r_i \notin S_0$. Then from Table 3, we have $r_i \in Inv(N)$, which implies that {all invest} is an equilibrium solution. From Table 1, the total cost for any agent in the equilibrium {all invest} equals C. There are two other possible cases for the number of agents M investing at equilibrium:

1. $M = 0$. In this case, from Table 1, we know that the total cost for agent i equals $L/\left[1 + r_i/\left(\lambda + (N-1)q\lambda\right)\right]$. By the definitions of $R_2(0)$, we have $C = L/\left[1 + R_2(0)/\lambda\right]$. Since we define $S_0 = (R_2(0), \infty)$ and we know $r_i \notin S_0$, we must have $R_2(0) \geq r_i$. So we will have $L/\left[1 + R_2(0)/\lambda\right] \leq L/\left[1 + r_i/\lambda\right] < L/\left[1 + r_i/\left(\lambda + (N-1)q\lambda\right)\right]$ for $N \geq 2$. Therefore, we have

$$C < L/\left[1 + r_i/\left(\lambda + (N-1)q\lambda\right)\right] \;\; \forall i = 1, ..., N \tag{6}$$

2. $1 \leq M \leq N-1$. In this case, from Table 1, we know that the total cost for an investing agent i equals $C + L/\left[1 + r_i/(N-M)\lambda\right]$, which is clearly greater than C; i.e.,

$$C < C + L/\left[1 + r_i/(N-M)\lambda\right] \;\; \forall i = 1, ..., M \tag{7}$$

   Similarly, the total cost for a non-investing agent i equals $L/\left[1 + r_i/\left(\lambda + (N-M-1)q\lambda\right)\right]$ for $i = M+1, ..., N$. So, as in case 1 above ($M = 0$), we have

$$C < L/\left[1 + r_i/\left(\lambda + (N-M-1)q\lambda\right)\right] \;\; \forall i = M+1, ..., N \tag{8}$$

Therefore, the total cost borne by each agent in the equilibrium {all invest} is lower than the corresponding cost in any other possible equilibrium solution. Finally, as N grows (assuming that all other parameters are held constant), straightforward algebra shows that the cost difference between the left and right sides of inequalities (6), (7), and (8) will increase. $\square$

9.4 Proof of Theorem 4

Theorem 1 indicates that investing and not investing are strictly dominant strategy for agent i if and only if $r_i = r \in S_N$ and $r_i = r \in S_0$, respectively. Since we assume that no agent has a strictly dominant strategy, we will have $r_i \notin [S_0 \cup S_N]$. Then from Table 3, we have $r_i \in Inv(N)$ and $r_i \in Non(0)$, which implies that {all invest} and {none invest} are both equilibrium solutions. Suppose that there exists an equilibrium solution with M agents having security such that $1 \leq M \leq N-1$. Then, according to the definition of equilibrium, we must have $r_i \in Inv(M)$ $\forall i = 1, ..., M$ and $r_i \in Non(M)$ $\forall i = M+1, ..., N$. However, homogeneity implies that $r_i = r$ $\forall i = 1, ..., N$, and we know that $Inv(M) \cap Non(M) = \emptyset$. Therefore, a contradiction has been found. $\square$

9.5 Proof of Theorem 5

If $h \leq N-k-1$, then after $k-1$ agents change from investing (in the sub-equilibrium $\{\text{all invest}\}_{N-h}$) to not investing, $\{\text{all invest}\}_{N-h-k+1}$ is still a sub-equilibrium for the remaining $N-h-k+1$ agents, because $r \in \mathbf{Cl}(S_k) \subset Inv(N-k+1)$. However, after k agents change from investing to not investing, $\{\text{all invest}\}_{N-h-k}$ is no longer a sub-equilibrium for the remaining $N-h-k$ agents, because $r \in \mathbf{Cl}(S_k) \not\subset Inv(N-k)$. So, $n = k-1$ is the largest number of agents that can change strategies such that the remaining agents will want to continue investing at sub-equilibrium. Therefore, $\{\text{all invest}\}_{N-h}$ has stability $\alpha = (k-1)/(N-h-1)$. Similarly, it can be shown that $\{\text{none invest}\}_{N-h}$ has stability $\alpha = (N-h-k-1)/(N-h-1)$ if $h \leq N-k-1$. Now we consider the case $h \geq N-k$. In this case, $\{\text{all invest}\}_{N-h}$ is a sub-equilibrium for the non-subsidized $N-h$ agents, and has stability $\alpha = 1$ (since for $h \geq N-k$, it is no longer possible to have k of the $N-h$ non-subsidized agents change strategies). To see why $\{\text{none invest}\}_{N-h}$ is not a sub-equilibrium in this case, note that if all of the $N-h$ non-subsidized agents choose not to invest, there will be only $M = h$ agents having security measures. From Table 3, we know that $\{\text{none invest}\}_{N-h}$ will be a sub-equilibrium for the $N-h$ non-subsidized agents if and only if $r \in \mathbf{Cl}(S_k) \subset Non(h)$, or equivalently $h \leq N-k-1$. □

9.6 Proof of Theorem 7

Since the agents are assumed to be homogeneous, by the similar argument as in Theorem 4, we know that all of the non-subsidized $N-h-x$ agents not making erroneous choices will choose the same strategy in any sub-equilibrium. If all of these $N-h-x$ agents choose to invest, then there will be a total of $M = N-x$ agents having security measures. From Table 3, we know that $\{\text{all invest}\}_{N-h-x}$ will be a sub-equilibrium for the $N-h-x$ non-subsidized agents not making erroneous choices if and only if $r \in \mathbf{Cl}(S_k) \subset Inv(N-x)$, or equivalently

$$x \leq k-1 \tag{9}$$

In this case, Table 1 indicates that the total cost borne by each of the agents receiving subsidized (free) security is given by $C_{Sub1} \equiv L/\left[1 + r/(xq\lambda)\right]$; the total cost borne by each of the non-subsidized $N-h-x$ agents not making erroneous choices is given by $C_{Inv} = C + L/\left[1 + r/(xq\lambda)\right]$, where $C > 0$ is the cost for any one agent to invest in security; and the total cost borne by each of the x agents making erroneous choices is given by $C_{Err} \equiv L/\left[1 + r/(\lambda + (x-1)q\lambda)\right]$.

Similarly, if all of the remaining $N-h-x$ agents choose not to invest, then there will be only $M = h$ agents having security measures. Again, from Table 3, $\{\text{none invest}\}_{N-h-x}$ will be a sub-equilibrium for these $N-h-x$ agents if and only if $r \in \mathbf{Cl}(S_k) \subset Non(h)$, or equivalently

$$h \geq N-k-1 \tag{10}$$

In this case, Table 1 shows that the total cost borne by each of the agents receiving free security is given by $C_{Sub2} \equiv L/\left[1 + r/\left((N-h)q\lambda\right)\right]$, and the total cost borne by any of the $N-h$ non-subsidized agents is given by $C_{Non} \equiv L/\left[1 + r/\left(\lambda + (N-h-1)q\lambda\right)\right]$.

There will always exist at least one sub-equilibrium, because at least one of inequalities (9) and (10) will hold (by the assumption that $0 \leq x \leq N-h$, and the fact that x, h, k, and N are all integers). If $h \geq N-k$, then $\{\text{none invest}\}_{N-h-x}$ will not be a possible sub-equilibrium solution, so $\{\text{all invest}\}_{N-h-x}$ will be the unique sub-equilibrium for all values of $x \leq N-h$. Conversely, if $x \geq k$, then $\{\text{all invest}\}_{N-h-x}$ will not be a sub-equilibrium, so $\{\text{none invest}\}_{N-h-x}$ will be the unique sub-equilibrium for all values of $h \leq N-k-1$. Finally, if $x+1 \leq k \leq N-h-1$, then both $\{\text{all invest}\}_{N-h-x}$ and $\{\text{none invest}\}_{N-h-x}$ will be sub-equilibrium solutions. In this case, straightforward algebra shows that we will have $C_{Inv} < C_{Non}$, $C_{Err} \leq C_{Non}$, and $C_{Sub1} \leq C_{Sub2}$. Thus, the costs borne by any of the N agents individually in the sub-equilibrium $\{\text{all invest}\}_{N-h-x}$ will be less than or equal to the corresponding costs in the sub-equilibrium $\{\text{none invest}\}_{N-h-x}$. □

9.7 Proof of Theorem 8

From Theorem 7, if $h \geq N-k$, then we must have $P_{Inv} = 1$, since $\{\text{all invest}\}_{N-h-x}$ is the unique sub-equilibrium for any value of x in that case. From inequality (9), if $h \geq N-k$, then the probability that $\{\text{all invest}\}_{N-h-X}$ is a sub-equilibrium is given by $P_{Inv} = P(X \leq k-1)$. □

9.8 Proof of Theorem 9

For the sub-equilibrium $\{\text{all invest}\}_{N-h-x}$ (if it exists), $C_x(h) = C_x(h-1)$. Therefore, for $h \geq 1$, we will have:

$$\begin{aligned} C_S(h) - C_S(h-1) &= C_F(h) + C_x(h) + C_O(h) - C_F(h-1) - C_x(h-1) - C_O(h-1) \\ &= [C_F(h) - C_F(h-1)] - \left[C + \frac{L}{1 + r/xq\lambda}\right] \\ &\leq C - C \\ &= 0 \end{aligned}$$

Therefore, we will have $C_S(h)$ non-increasing in h in this case. For the sub-equilibrium $\{\text{none invest}\}_{N-h-x}$ (if it exists), similar to the proof of Theorem 3, we can show that $r \leq R_2(0)$, or equivalently, $C - L/(1 + r/\lambda) \leq 0$. Therefore, for $h \geq 1$, we will have:

$$\begin{aligned} C_S(h) - C_S(h-1) &= C_F(h) + C_x(h) + C_O(h) - C_F(h-1) - C_x(h-1) - C_O(h-1) \\ &= (C_F(h) - C_F(h-1)) + \frac{(N-h)L}{1 + r/[\lambda + (N-h-1)q\lambda]} - \frac{(N-h+1)L}{1 + r/[\lambda + (N-h)q\lambda]} \\ &\leq C + \frac{(N-h)L}{1 + r/[\lambda + (N-h)q\lambda]} - \frac{(N-h+1)L}{1 + r/[\lambda + (N-h)q\lambda]} \\ &= C - \frac{L}{1 + r/[\lambda + (N-h)q\lambda]} \\ &\leq C - \frac{L}{1 + r/\lambda} \\ &\leq 0 \end{aligned}$$

Therefore, we will have $C_S(h)$ non-increasing in h in this case. $\square$