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Assessing Joint Distributions with Isoprobability Contours

Ali E. Abbas
Department of Industrial and Enterprise Systems Engineering, College of Engineering, University of Illinois at Urbana–Champaign, Urbana, Illinois 61801, aliabbas@uiuc.edu

David V. Budescu
Department of Psychology, Fordham University, Bronx, New York 10458; and Department of Psychology, University of Illinois at Urbana–Champaign, Urbana, Illinois 61801, budescu@fordham.edu

Yuhong (Rola) Gu
Department of Psychology, University of Illinois at Urbana–Champaign, Urbana, Illinois 61801, gyh1224@gmail.com

We present a new method for constructing joint probability distributions of continuous random variables using isoprobability contours—sets of points with the same joint cumulative probability. This approach reduces the joint probability assessment into a one-dimensional cumulative probability assessment using a sequence of binary choices between various combinations of the variables of interest. The approach eliminates the need to assess directly the dependence, or association, between the variables. We discuss properties of isoprobability contours and methods for their assessment in practice. We also report results of a study in which subjects assessed the 50th percentile isoprobability contour of the joint distribution of weight and height. We use the data to show how to use the assessed contours to construct the joint distribution and to infer (indirectly) the dependence between the variables.

Key words: isoprobability contours; joint probability elicitation; probability encoding; correlation; dependence

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1. Introduction

The construction of representative probability distributions from a decision maker (DM) is a fundamental step in decision analysis and has engendered a substantial literature. It is well known that the probability elicitation process for simple events or continuous random variables can be subject to many cognitive and motivational biases (see, for example, Hogarth 1987, Kahneman and Tversky 1973, Spetzler and Staël von Holstein 1975, Tversky and Kahneman 1974, Wallsten and Budescu 1983). When eliciting a joint probability distribution, the magnitude of this task is much larger because we are faced with the need to incorporate the dependence among the random variables into the analysis. Several authors have discussed these issues and have presented methods to facilitate the elicitation process (see O’Hagan et al. 2006 for a review). For example, Howard (1989) proposed evocative and redundant knowledge maps to graphically represent the dependence relations and provided examples of how they simplify the construction of joint probability distributions. Other methods approximate the joint distribution in terms of lower-order joint probability assessments. For example, Keefer (2004) presented a binary event model for approximating dependence relations between the variables, and Abbas (2006) presented a maximum entropy approach for constructing a discrete joint probability distribution using lower-order joint probability assessments. Some methods for constructing joint probability distributions of continuous variables require information about the cross-moments or the correlation coefficients between the variables. For example, Cooke (1991) proposed a continuous maximum entropy joint distribution using moments and correlation coefficients, Yi and Bier (1998) constructed joint distributions using copula structures and certain dependence parameters for risk analysis, and Clemen and Reilly (1999) used multivariate normal copulas to construct a joint distribution using pairwise correlation coefficients.

Several “direct” methods for assessing dependence between variables have also been proposed. For example, Gokhale and Press (1982) elicited concordance probabilities from subjects. Under the assumption of
bivariate normality, these probabilities can be related to the Pearson correlation coefficient.\footnote{Pr(x < μx, y < μy) = 0.25 + sin^{-1}(ρ_{xy}/2\pi).} Clemen et al. (2000) conducted a study comparing various methods of eliciting the correlation between pairs of variables. Under the assumption of bivariate normality, it is possible to use these estimates to infer the joint probability distribution of the two variables. They found that direct assessments of correlation coefficients on a scale of −1 to 1 outperformed several other methods. This result contradicts previous studies (such as Morgan and Henrion 1990, Kadane and Wolfson 1998).

The judgment literature shows that people are notoriously poor estimators of association and correlation between variables. For example, Meyer and Shinar (1992) and Meyer et al. (1997) studied the inference of correlations based on visual displays (scatterplots) and found that the dispersion of the data and the presence of outliers greatly affected the perception of the correlation and led to large underestimation of the correlation between the variables. Other researchers (e.g., Alloy and Tabachnik 1984, Arkes and Harkness 1983, Beyth-Marom 1982, Hamilton and Gifford 1976, Trolier and Hamilton 1986) showed that judgments of covariation and correlations are highly sensitive to (a) the method and format of data presentation, (b) the instruction for judgments, and (c) the judges’ prior expectations. In particular, people tend to detect “illusory correlations,” i.e., systematic covariation in cases where it is not present (e.g., Allan and Jenkins 1980, Chapman 1967).

In this paper, we present a new and fundamentally different method for constructing joint probability distributions of continuous random variables. This approach dispenses with the need to assess conditional probability distributions, correlation coefficients, concordance probabilities, or conditional percentiles. The proposed procedure reduces the joint probability assessment into a one-dimensional marginal probability assessment of one of the variables, and the elicitation of one or more “isoprobability contours.”

An isoprobability contour of a joint cumulative distribution is the collection of all the points that have the same cumulative probability, and is analogous to an isopreference contour for a multiattribute utility function (Keeney and Raiffa 1976). Although there has been work on assessing isopreference contours (e.g., the classic paper by MacCrimmon and Toda (1969) on assessing isopreference contours of bundles of money and ballpoint pens), and work by Matheson and Abbas (2005) on deriving the relation between conditional utility functions of attributes using isopreference contours and a one-dimensional utility function over value, we are not aware of theoretical or empirical work on assessing isoprobability contours of joint cumulative distributions. As we show next, isoprobability contours have some additional properties that facilitate their assessment in practice.

The differences between the isopreference and isoprobability contours are due to the different levels of measurement of the utilities (typically, interval scales unique up to a positive linear transformation) and probabilities (absolute scales). Thus, this work goes beyond simple reparameterization of the isopreference contours.

We also propose a method to assess isoprobability contours in practice. This approach does not require the subjects to provide any numeric judgments, but simply to state their preferences over binary lotteries having identical outcomes. The approach can therefore be used with judges having less technical expertise in probability or statistics. The probabilities defining the two lotteries involve the two variables of interest and, as such, cause the judge to consider the relevant trade-offs between them. To the best of our knowledge, this is the only method that treats these probability trade-offs directly and explicitly and makes them an integral part of the elicitation process. Once the isoprobability contours are determined, a one-dimensional marginal probability assessment over any of the variables (or over the contours themselves) is sufficient to determine the complete joint distribution of all the variables. As a result, isoprobability contours can also be used to determine the dependence parameters between the variables of the decision situation.

The remainder of this paper is structured as follows. Section 2 presents some properties of isoprobability contours that facilitate their elicitation. Section 3 presents methods to assess isoprobability contours and methods to construct the joint distribution. Section 4 describes the experimental procedure. Section 5 reports results of a study designed to test the feasibility of assessing isoprobability contours. Section 6 presents methods to curve fit isoprobability contours to functional forms to simplify the construction of the joint distribution. Section 7 curve fits isoprobability contours to a bivariate Gaussian copula and estimates the dependence between the two variables. Section 8 summarizes our results.

## 2. Properties of Isoprobability Contours

To start, consider two continuous variables, X and Y, having a joint cumulative probability distribution, \(F(x, y)\), over a connected bounded domain, i.e., \(x \in [x_{\text{min}}, x_{\text{max}}], y \in [y_{\text{min}}, y_{\text{max}}]\). An isoprobability contour function, \(C_i(x, y)\), is the set of all points in the \(x\)-\(y\) plane that have the same cumulative probability, \(C_i\):

\[
C_i(x, y) = \{(x, y): F(x, y) = C_i\}. \tag{1}
\]
Figure 1  Joint Cumulative Distribution and Isoprobability Contours

Figure 1 shows an example of a joint cumulative distribution and the corresponding two-dimensional projection of some of its isoprobability contours.

Isoprobability contours have several properties that can facilitate their elicitation in practice and can serve as consistency checks during the elicitation process. We discuss some of these properties below.

**Property 1.** The left and bottom boundaries of the domain are isoprobability contours.

**Proof.** A joint cumulative distribution has a value of zero if any of the variables is at the minimal value of the domain. This is known as the “grounding property,” where

\[ F(x_{\min}, y_{\min}) = F(x_{\max}, y_{\max}) = 0 \]

\[ \forall x \in [x_{\min}, x_{\max}], \ y \in [y_{\min}, y_{\max}] \]  \hspace{1cm} (2)

Equation (2) implies that all points on the axes \( x = x_{\min} \) and \( y = y_{\min} \) have a joint cumulative probability of zero and therefore lie on the same isoprobability contour. \( \square \)

**Property 2.** If differentiable, isoprobability contours have a nonpositive slope.

**Proof.** Recall that the change in probability along an isoprobability contour must be zero. Hence,

\[ dF(x, y) \bigg|_{C_i} = \left[ \frac{\partial F(x, y)}{\partial x} \right] dx + \left[ \frac{\partial F(x, y)}{\partial y} \right] dy = 0 \]  \hspace{1cm} (3)

Rearranging (3) gives

\[ \frac{dy}{dx} \bigg|_{C_i} = -\frac{\partial F(x, y)/\partial x}{\partial F(x, y)/\partial y} = -\frac{F_y(y)(\partial F_{x\mid y}(x \mid y)/\partial x)}{F_x(x)(\partial F_{y\mid x}(y \mid x)/\partial y)} \]  \hspace{1cm} (4)

where \( F_{x\mid y}(x \mid y) \triangleq P(X \leq x \mid Y \leq y) \). The right-hand side of (4) is nonpositive, because a cumulative distribution function (CDF) is nonnegative and nondecreasing. \( \square \)

Given some assessed points on an isoprobability contour, Property 2 can be tested by calculating Kendall’s \( \tau_b \) rank-order correlation between the points, as we illustrate in \( \S 5 \).

The following properties are also useful and follow directly from the definition of isoprobability contours.

**Property 3.** Isoprobability contours do not intersect.

**Property 4.** Isoprobability contours connect points with the same marginal probability, as shown in Figure 2.

In the next section we use these properties to assess isoprobability contours and to verify the quality of the assessments.
3. Assessing an Isoprobability Contour and Constructing the Joint Distribution

3.1. Assessing an Isoprobability Contour
Suppose we have assessed the median (50th percentile) of variable $X$, which we denote as $x_{50}$, as well as other fractiles of $X$, such as $x_{75}$ and $x_{90}$. We now wish to assess the 50% isoprobability contour. We start by offering the DM a choice between two deals with identical outcomes. Thus, the DM’s choice should depend solely on the probabilities. Consider for example, the following two deals.

**Deal A:** The DM receives a fixed amount, $s$, if the outcome of variable $X$ is less than $x_{50}$ and variable $Y$ takes any value (i.e., $Y \leq y_{\text{max}}$).

**Deal B:** The DM receives the same fixed amount, $s$, if the outcome of variable $X$ is less than $x_{75}$ and the outcome of variable $Y$ is less than $y_1$ (where $y_1 < y_{\text{max}}$).

Based on the DM’s response, we adjust the value of $y_1$ sequentially until the user expresses indifference between $(x_{50}, y_{\text{max}})$ and $(x_{75}, y_1)$. If the DM does not express indifference between the two deals after a predetermined number of binary questions, we settle for upper and lower bounds for the value of $y_1$, which we denote as $(x_{75}, y_1^\text{L})$ and $(x_{90}, y_1^\text{U})$, respectively, so we may have a narrow band around the contour. We consider the midpoint of this band as the estimate of $y_1$. We then move on to the next point and ask the DM to compare deals based on $(x_{75}, y_1)$ and $(x_{90}, y_2)$ (where $y_2 < y_1$), and change $y_2$ in a similar fashion until we reach indifference between them. Through this chaining process we trace the 50% isoprobability contour of the DM. This process is illustrated in Figure 3. It is possible to supplement this process with consistency checks involving previously elicited points lying on the contour.

This process is symmetric across variables, i.e., the roles of the X’s and Y’s can be reversed.

3.2. Probability Trade-Offs
For any two points $(x_1, y_1)$ and $(x_2, y_2)$ along the same isoprobability contour, consider the ratio

$$\frac{\Delta y_2}{\Delta x_2} \bigg|_{Q} = \frac{y_2 - y_1}{x_2 - x_1}. \quad (5)$$

This ratio is nonpositive (Property 3) and represents the increase in the value of one variable needed to compensate for a decrease in another to achieve the same joint probability. As the increment in one variable decreases (and $\Delta x \to 0$), this ratio approaches the slope of the isoprobability contour (the probability trade-off). This provides a simple and intuitive interpretation for the slope of the isoprobability contours—the change in one variable needed to compensate for a change in the other, to achieve the same cumulative probability. We can also define the ratio of percentage change in one variable needed to compensate a percentage change in the other,

$$\eta(x, y) = -\frac{\Delta y/y}{\Delta x/x} \bigg|_{Q}. \quad (6)$$

We have chosen to add a negative sign in this definition to yield nonnegative values of the probability trade-off. This ratio is easier to think about than the absolute value of the increments because it is dimensionless and the DM needs only to provide a percentage between 0% and 100%. As the increment $\Delta x \to 0$ in (6), a fractional change corresponds to the derivative, $\eta = -(dy/dx)(x/y)$. Contours having a constant $\eta$ define a family of functional forms whose shape depends on this parameter and are often referred to as the constant elasticity of substitution curves.

3.3. Constructing the Joint Probability Distribution
Once the isoprobability contours have been determined, the construction of the joint distribution

Figure 3 Assessing Points on an Isoprobability Contour by Indifference Choices
requires only the assessment of one of the marginal probability functions, say $F_x(x)$. We can determine the joint cumulative distribution of any point, $(x, y) \in [x_{\min}, x_{\max}] \times [y_{\min}, y_{\max}]$, by finding the point $(x_1, y_{\max})$ that lies on its isoprobability contour. By definition,

$$F(x, y) = F(x_1, y_{\max}) \equiv F_x(x_1).$$ (7)

This formulation reduces the joint probability into a one-dimensional marginal probability. The marginal distribution function, $F_x$, can be assessed directly using any of the standard techniques of probability encoding used in decision analysis (Spetzler and Staël von Holstein 1975, von Winterfeldt and Edwards 1986, Abbas et al. 2008), or by assuming a particular functional form (e.g., a beta distribution) and assessing its parameters by assessing a few probability points using indifference to binary lotteries. This process is symmetric across variables, i.e., we can replace $F_x$ by $F_y$.

Define

$$x_1(x, y) = x_1 \text{ such that } C(x, y) = C(x_1, y_{\max}).$$

For any $(x, y)$, define $x_1$ such that $C(x, y) = C(x_1, y_{\max})$. Then, $F(x, y) \equiv F_x(x_1)$, and the conditional probability, $F(x \mid y)$, is

$$F(x \mid y) = \Pr(X \leq x \mid Y \leq y) = \frac{F(x, y)}{F_y(y)} = \frac{F_x(x_1)}{F_y(y)}. \quad (8)$$

The conditional probability functions at any point in the plane can also be related explicitly to the assessed trade-off functions of (55) and the marginal distributions using the same transversality relations that were developed in Matheson and Abbas (2005).

### 4. An Empirical Study

#### 4.1. Design

The goal of this study is to determine whether the judges can make sensible probability trade-off judgments that can be used to infer meaningful joint distributions. We asked a number of judges to assess trade-off functions of (55) and the marginal distributions and then assessed the contour of the joint distribution.

#### 4.2. Eliciting the Fractiles of the Marginal Distributions

To elicit the marginal probability distributions of the height and weight variables, we used pairs of gambles involving (a) a probability wheel and (b) one event related to the weight or the height of a randomly selected male student from UIUC. For example, a wheel showing a 50% chance of winning was compared with the possibility that the weight of a randomly selected male student at UIUC will be less than 120, 140, 160 lbs, etc.

In the face-to-face groups, we used the fixed variable (FV) method for eliciting the fractiles of the marginal distributions, where the variable values are fixed and the wheel setting is adjusted based on the responses obtained (for more information on this approach, see...
Spetzler and Staël von Holstein 1975, Abbas et al. 2008). Subjects were presented with five weights or heights which they compared with various settings of the probability wheel until they reached (or were close to) indifference between the two options. We elicited between 4 and 12 points for each distribution based on the experimenter’s judgment. The order of elicitation of the two variables was randomized.

In the computerized group, we used the fixed probability (FP) method for eliciting the fractiles of the marginal distributions (see Spetzler and Staël von Holstein 1975, Abbas et al. 2008). With this approach, the probability wheel setting is fixed and corresponding values of the variables are elicited. We used five fixed settings of the wheel (5%, 25%, 50%, 75%, and 95%), which were compared with various weights or heights until the judges reached (or were close to) indifference between the two options. Figure 4 presents an example of a display for the marginal distribution elicitation task.

Each point was based on a sequence of interrelated choices. The subject’s choice at step \( t \) was used to determine the pair shown on step \( (t + 1) \), which was always selected to make the two options more similar (i.e., closer to indifference). In the face-to-face group, we asked questions until subjects reached indifference. In the computerized group, a series was terminated when the subject expressed indifference between the two options, or when the number of comparisons exceeded six. Thus, in some cases we ended with a range of values, rather than a point. Overall, subjects reached indifference in 54% of the judgments. At the end of each series, subjects were presented with a “prediction” of their preferences (based on their previous choices), and were asked to confirm it before moving to the next series.

4.3. Eliciting the Isoprobability Contours of the Joint Distribution

To elicit isoprobability contours of the joint distribution the subjects (in both groups) were asked to consider pairs of events related to the weight and the height of randomly selected male students at UIUC. In each pair, one of the events was fixed but the description of the other event changed. For example, the event of choosing a UIUC student “whose height is less than 6’1” and who weighs less than 148 lbs” was compared with the event of selecting a UIUC student “whose height is less than 6’2” and who weighs less than X lbs,” where the weight varied from one stage to the next (for example, \( X \) could be 112, 125, or 140 lbs, etc.). Subjects performed comparisons with various weights (values of \( X \) ) in the second lottery until they reached (or were close to) indifference between the two options. Figure 5 presents an example of such a trial.

We obtained two series of five points on the 50% isoprobability contour—once by fixing the weight and another by fixing the height. In the fixed height (weight) series, subjects performed comparison with various weights (heights) of the second lottery while the height (weight) of the second lottery was fixed. For example, in a fixed weight case, the decision maker was offered a choice to bet on (a) the probability of choosing a UIUC student “whose height is less than 5’9” and who weighs less than 150 lbs” and (b) the chance of selecting a UIUC student “whose weight is less than 160 lbs and whose heights is less than \( x \) inches,” and the height varied from one trial to the next. For the fixed weight (height) series, the initial option was a student with “any height (weight)” and “weight (height) less than the value corresponding to a marginal cumulative probability of 50% elicited by the subject in the marginal section.”

In the face-to-face group, we asked questions until subjects reached indifference. In the computerized group, a series was terminated when the subject expressed indifference between the two options, or when the number of comparisons exceeded six. Subjects reached indifference in 59% of the cases.
There were wide individual differences in this respect. For instance, 3 of the 15 subjects reached indifference on all series, and 3 other subjects never expressed indifference. In the face-to-face setting, subjects had an extra pause (of about five minutes) between the marginal and each of the joint assessments.

At the end of the experiment, one pair of gambles was selected and was played according to the subject’s preference to determine the payment.4 Half the subjects won the lotteries. In the face-to-face group we also asked the subjects to rate the difficulty of the estimations tasks and to provide a direct estimate of the correlation they perceive between height and weight. Participants took between 50 to 60 minutes to complete all stages of the study.

5. Experiment Results
We now present the results of both groups. In those cases where we wish to highlight individual results, we focus on the face-to-face group where all subjects reached indifference on all series. Figure 6 shows the joint assessments for the 10 subjects in the face-to-face group (all converted to centimeters and kilograms). For each subject, we plot all the points that were elicited using the two sequences (fixed weights and fixed heights) jointly, but we use different symbols to identify the two.

5.1. Monotonicity
Our first analysis is concerned with the monotonicity of the assessments. We use Kendall’s τb as a measure of monotonicity. For any n assessments on a contour, let C be the number of points that are concordant (i.e., an increase in weight corresponds to an increase in height and the monotonicity condition of Property 2 is violated), and let D be the number of pairs that are discordant (i.e., the monotonicity condition of Property 2 is satisfied). Kendall’s τb is the difference between the proportion of concordant pairs and the proportion of discordant pairs. Formally,5

\[
\tau_b = \frac{\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \text{sgn}(\text{height}_i - \text{height}_j) \text{sgn}(\text{weight}_i - \text{weight}_j)}{\binom{n}{2}}
\]

\[
= \frac{C - D}{\binom{n}{2}} = \frac{C - D}{C + D'}
\]

(9)

where sgn is the sign function.

4 Once a pair of gambles was chosen and the subject’s response was identified, we compared it with the weight and height of a “randomly selected undergraduate student.” This undergraduate student was in fact the programmer for the experiment. The weight and height of this undergraduate student were fixed in advance in the program to determine the payoff.

5 In the presence of ties, the denominator of the formula is \(\sqrt{(C + D + T_x)(C + D + T_y)}\), where \(T_x\) is the number of pairs with ties on X (but not on Y), and \(T_y\) is the number of pairs with ties on Y (but not on X).

Kendall’s τb is a measure of rank-order correlation that ranges from −1 (all pairs are discordant) to 1 (all pairs are concordant), and it is 0 when there are equal numbers of concordant and discordant pairs.

For comparison purposes, we also present measures of monotonicity for the two marginal distributions (in this case, perfect monotonicity implies τb = 1). Table 1 shows the median Kendall’s τb for each distribution for both groups. All of the (absolute) values are very high, confirming the monotonicity of the judgments for the marginal distributions and the isoprobability contours. They are highly similar in all cases, indicating that subjects were equally adept in making the univariate and the joint assessments.6

5.2. Estimating the Implied Probability Trade-Offs
For each two adjacent points, \((x_i, y_i)\) and \((x_j, y_j)\), on the assessed contour, we calculated the increments \(\Delta y_i = y_j - y_i\) and \(\Delta x_i = x_j - x_i\), and then the ratio

\[
\eta_i = \frac{\Delta y_i / y_i}{\Delta x_i / x_i} = \frac{\Delta \text{height} / \text{height}}{\Delta \text{weight} / \text{weight}}
\]

(see Equation (6)). We then calculated for each subject a robust measure of the spread of the \(\eta_i\) ratios—the median absolute deviation (MAD) around their median value, MAD(\(\eta_i\)), across the various points on the assessed contour. A constant probability trade-off implies \(\text{MAD}(\eta_i) = 0\). In the face-to-face group, the individual MADs range from 0.05 to 0.77, with a median of 0.19. In the computerized group, we have highly similar results—the 15 MADs range from 0.04 to 0.73, with a median value of 0.24. These values suggest that for many subjects, a constant probability trade-off provides a reasonable first-order approximation of the contour function.

5.3. Internal Consistency
Recall that we assessed the 50th percentile isoprobability contour in both directions: On one occasion we started with the median value of the weight (obtained from the marginal distribution assessments)

4 In the face-to-face group, we also asked the subjects to rate how easy it was to assess the marginal and the joint distributions to compare the perceived difficulty of the two tasks. The split was 6/4 (which is not significantly different from chance, by a sign test), indicating that, on average, the two tasks are equally easy.

---

**Table 1** Median Kendall’s τb, Rank-Order Correlation for the Various Distributions in the Two Groups

<table>
<thead>
<tr>
<th>Group</th>
<th>Sample size</th>
<th>Marginal distributions</th>
<th>Joint distributions</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Height</td>
<td>Weight</td>
<td>Height fixed</td>
</tr>
<tr>
<td>Computerized</td>
<td>15</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>Face-to-face</td>
<td>10</td>
<td>0.99</td>
<td>0.90</td>
</tr>
<tr>
<td>All</td>
<td>25</td>
<td>1.00</td>
<td>1.00</td>
</tr>
</tbody>
</table>

[1003]
with no constraints on the height, and then offered deals with increased weights and reduced heights. We repeated the assessment starting with the median height (no constraints on the weight), and then offered deals with increasing heights and decreasing weights. Consistency with Property 4 requires that the assessed contours connect the median values of each variable, i.e., the median height assessed from the marginal distribution, for example, should equal the value of the height intersecting the 50% contour assessment with the vertical axis at the upper bound of the weight variable (Figure 2).
Figure 7 displays these consistency results for the 10 subjects in the face-to-face group. For each variable, we plot the median of the variable obtained from the assessment of its marginal distribution and the median of the same variable as inferred from the contour assessment. Perfect consistency requires that all points would lie on the identity line. In addition to the identity line, the plots include narrow bands of ±5 units (centimeters for height and kilograms for weight), and each subject is identified by his/her number. Note that most points are, indeed, within this narrow band and close to the ideal with only two exceptions (weight judgments of Subject 1 and height judgments for Subject 10). We conclude that the observed consistency between the assessed and inferred medians is impressively high for both variables.

Figure 8 shows, as an example, the isoprobability contours implied by the Cobb-Douglas contour functions $C(x, y) = x^\alpha y^\gamma$ for $\eta = 0.8$ in the left panel, and $\eta = 2.4$ in the right panel. The contours in both panels connect the marginal probability values of each variable starting at $(x_{\text{min}}, y)$ and ending at $(x, y_{\text{max}})$. The marginal distribution of $Y$ is the same for both cases but, as the value of $\eta$ increases, the contours end at points $(x, y_{\text{max}})$ with higher values of the variable $X$. This models a joint distribution whose marginal probability mass for variable $X$ is concentrated at higher values.

In many cases, the assumption that $\eta$ is constant can provide a good first-order approximation, and it is convenient, because it determines the functional form of the contours in terms of a single parameter. Re-arranging terms in (10) gives

$$\log(x - x_{\text{min}}) = \log C(x, y) - \eta \log(y - y_{\text{min}}).$$

This is easily recognized as a linear regression equation, and the parameters can be estimated by ordinary least squares (OLS) regression. In particular, the slope estimates the (negative) probability trade-off coefficient. We illustrate this approach for one subject (Subject 4 in the face-to-face group).

6. Some Functional Forms of Isoprobability Contours

The experimental results indicate that people can reason about the isoprobability contours. To simplify their assessment, one could assume a particular functional form for the contours and assess its parameters. Of course, the choice of the functional form depends on the qualitative properties of the contours provided by the DM. For example, suppose that for a given DM we assume that the percentage increase in one variable needed to compensate for a percentage decrease in the other to achieve the same cumulative probability is constant. This implies that the value of $\eta$, is constant ($\eta$), and the DM’s subjective contours can be described by constant elasticity of substitution functions of the form

$$C(x, y) = (x^\alpha + \eta y^\gamma)^{1/\alpha}, \quad 0 < a < 1,$$

where $C(x, y)$ is a contour function, special cases of which are the Cobb-Douglas-type contour functions $C(x, y) = x^\alpha y^\gamma$. In the left panel, and $\eta = 2.4$ in the right panel. The contours in both panels connect the marginal probability values of each variable starting at $(x_{\text{max}}, y)$ and ending at $(x, y_{\text{max}})$. The marginal distribution of $Y$ is the same for both cases but, as the value of $\eta$ increases, the contours end at points $(x, y_{\text{max}})$ with higher values of the variable $X$. This models a joint distribution whose marginal probability mass for variable $X$ is concentrated at higher values.

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7 We selected this ±5 margin to reflect the natural rounding tendencies of most people in reporting heights and weights.

8 Cobb-Douglas-type functions are a special case of the constant elasticity of substitution family when we take the limit when $\alpha \to 0$ and using L’hopital’s rule.
To illustrate this approach, we present in Figure 9 the elicited points of one of our subjects and the fitted function on a log-log scale, as well as the parameters estimated by OLS. The figure shows the estimated probability trade-off coefficient obtained by fixing the weight, fixing the height, and by using both assessment methods for the estimation. The linear function fits the data points well in all three panels (which yield highly similar parameters), suggesting a constant probability trade-off, and supports the choice of the one-parameter function used in this case.

Once the value of \( \eta \) is estimated, the contour function is fully determined and the joint cumulative probability at any point \((x, y)\) can be obtained by determining the boundary value using either marginal distribution:

(i) Using the marginal distribution of \( X \), calculate the value of \( x_1 \) that corresponds to \((x - x_{\min}) (y - y_{\min})^\eta = (x_1 - x_{\min})(y_{\max} - y_{\min})^\eta \) and determine the cumulative probability of \( x_1 \) using its marginal distribution, \( F_x(x_1) \).

(ii) Using the marginal distribution of \( Y \), calculate the value of \( y_1 \) that corresponds to \((x - x_{\min}) (y - y_{\min})^\eta = (x_{\max} - x_{\min})(y_1 - y_{\min})^\eta \) and determine the cumulative probability of \( y_1 \) using its marginal distribution, \( F_y(y_1) \).

We get \( x_2 = 68.42 \) kg, \( y_2 = 177.57 \) cm. The joint cumulative probability of \((88.45 \) kg, \( 177.8 \) cm) is equal to the marginal cumulative probability of \( weight \leq 68.42 \) kg, and the marginal cumulative probability of \( height \leq 177.57 \) cm. From the assessed marginal distributions we estimate \( F_x(68.42) = 0.43 \) and \( F_y(177.57) = 0.45 \), which are very similar.

### 6.2. Sensitivity Analysis

Because we need only one marginal distribution and a contour function to calculate the joint probability, consistency requires that calculations obtained using either marginal distribution should provide the same result. Figure 10 plots the joint probability of the point \((x = 88.45 \) kg, \( y = 177.8 \) cm) obtained using both marginal distributions (as we did in the previous example) for different values of \( \eta \). The figure shows several interesting results. First, note that when the estimate of \( \eta \) is between 1.38 and 1.66 (the values estimated from the two sets in Figure 9), there is little change in the joint cumulative probability using either marginal distribution. In other words, the two probability trade-offs (\( \eta = 1.38, \eta = 1.66 \)) yield highly similar joint probabilities. Perfectly consistent assessments (for both marginal distributions and isoprobability contours) occur at \( \eta = 1.32 \) (where the curves cross).

If we change the value of \( \eta \) while keeping the marginal distributions fixed, however, we get larger discrepancies. The explanation is that for any given value of \( \eta \) and one fixed marginal distribution, say of \( X \), we completely characterize the joint distribution, and we are determining the marginal distribution for variable \( Y \) (refer back to Figure 8, which shows how the contours connect different values of variable \( Y \) as we vary \( \eta \)).
7. Inferring Dependence from the Assessed Contours

Once the joint distribution is specified, we can determine any measure of association between the variables or any concordance probability by direct calculation. Isoprobability contours can also be used to infer dependence indirectly by curve fitting an appropriate joint probability model. For example, if the assessed curves fit a bivariate Gaussian distribution (or bivariate Gaussian copula), the estimated parameters of the curve fit will in fact be the correlation coefficients. Figure 11 shows the 50% isoprobability contours for several bivariate standard (i.e., \( \mu_x = \mu_y = 0 \) and \( \sigma_x = \sigma_y = 1 \)) Gaussian distributions. The seven contours correspond to distributions with different correlation coefficients, \( \rho \). The purpose of this section is to show how to determine the correlations coefficients between the variables if the assessed isoprobability contours match those of a bivariate normal copula. Similar procedures can be used with other families of joint distributions if the assessed contours do not match those of the Gaussian distribution.

7.1. Estimation of Correlation Coefficient by Curve Fitting to a Bivariate Gaussian Copula

To infer the correlation coefficient implied by the assessed isoprobability contour we curve fitted the data to a bivariate Gaussian copula. The curve-fitting procedure is as follows:

1. Determine the marginal distributions by fitting the elicited marginal fractiles into functional forms of cumulative distributions for weight and height, \( F_w(w) \) and \( F_h(h) \), respectively. We fitted beta distributions to the marginal fractiles because they allow a lot of flexibility in matching a variety of fractile assessments for bounded distributions. However, one could also use other distributions depending on their goodness of the fit.

2. Using the fitted marginal distributions, transform the elicited weight and height points \( (w, h) \) on the assessed contour into corresponding points on standardized normal distributions \( (w_N, h_N) \) by

\[
(w_N, h_N) = (\phi^{-1}(F_w(w)), \phi^{-1}(F_h(h))),
\]

where \( \phi \) is the standard normal distribution.
Table 2  Estimated Correlations Between Height and Weight and the Goodness of Fit (RMSE) of the Estimates

<table>
<thead>
<tr>
<th>Subject</th>
<th>RMSE</th>
<th>ρ</th>
<th>( (\rho_x = 6/\pi \sin^{-1}(\rho/2) )</th>
<th>( \tau_b = (2/\pi) \sin^{-1}(\rho) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) Estimates from face-to-face group</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.17</td>
<td>−0.48</td>
<td>−0.46</td>
<td>−0.32</td>
</tr>
<tr>
<td>2</td>
<td>0.10</td>
<td>0.80</td>
<td>0.79</td>
<td>0.59</td>
</tr>
<tr>
<td>3</td>
<td>0.06</td>
<td>0.39</td>
<td>0.37</td>
<td>0.25</td>
</tr>
<tr>
<td>4</td>
<td>0.03</td>
<td>0.92</td>
<td>0.91</td>
<td>0.74</td>
</tr>
<tr>
<td>5</td>
<td>0.07</td>
<td>0.89</td>
<td>0.88</td>
<td>0.70</td>
</tr>
<tr>
<td>6</td>
<td>0.21</td>
<td>0.89</td>
<td>0.88</td>
<td>0.70</td>
</tr>
<tr>
<td>7</td>
<td>0.13</td>
<td>0.69</td>
<td>0.67</td>
<td>0.48</td>
</tr>
<tr>
<td>8</td>
<td>0.08</td>
<td>0.22</td>
<td>0.21</td>
<td>0.14</td>
</tr>
<tr>
<td>9</td>
<td>0.06</td>
<td>0.88</td>
<td>0.87</td>
<td>0.68</td>
</tr>
<tr>
<td>10</td>
<td>0.07</td>
<td>0.99</td>
<td>0.99</td>
<td>0.91</td>
</tr>
<tr>
<td>Median</td>
<td>0.08</td>
<td>0.84</td>
<td>0.83</td>
<td>0.64</td>
</tr>
<tr>
<td>(b) Estimates from computerized group</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.74</td>
<td>0.99</td>
<td>0.99</td>
<td>0.91</td>
</tr>
<tr>
<td>2</td>
<td>0.28</td>
<td>0.83</td>
<td>0.82</td>
<td>0.62</td>
</tr>
<tr>
<td>3</td>
<td>0.31</td>
<td>−0.74</td>
<td>−0.72</td>
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<tr>
<td>4</td>
<td>1.71</td>
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<td>0.99</td>
<td>0.91</td>
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<tr>
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<tr>
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<td>0.71</td>
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<td>0.79</td>
<td>0.59</td>
</tr>
<tr>
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<td>0.39</td>
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<td>0.49</td>
<td>0.34</td>
</tr>
<tr>
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<td>0.07</td>
<td>−0.17</td>
<td>−0.16</td>
<td>−0.11</td>
</tr>
<tr>
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<td>0.04</td>
<td>0.70</td>
<td>0.68</td>
<td>0.49</td>
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<tr>
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<td>0.36</td>
<td>0.35</td>
<td>0.23</td>
</tr>
<tr>
<td>14</td>
<td>0.04</td>
<td>−0.05</td>
<td>−0.05</td>
<td>−0.03</td>
</tr>
<tr>
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<td>0.35</td>
<td>0.60</td>
<td>0.58</td>
<td>0.41</td>
</tr>
<tr>
<td>Median</td>
<td>0.28</td>
<td>0.70</td>
<td>0.68</td>
<td>0.49</td>
</tr>
</tbody>
</table>

(3) Determine the correlation between weight and height by curve fitting the transformed points \((w_{xy}, h_{xy})\) on the isoprobability contour to a bivariate Gaussian distribution. We searched for the value of \(\rho\) that minimizes the root mean square error (RMSE) between the standardized (transformed) points on the empirical 0.5 contour and the 0.5 contour of the bivariate Gaussian distributions (see examples in Figure 11). We searched over the range −0.99 to 0.99 in increments of 0.01.

Table 2 shows the estimated correlation coefficients and the RMSEs for all the subjects in both groups. The number of assessed points underlying this estimation was 10 for all the subjects in the computerized group. The number of points assessed in the face-to-face group varied from one person to another because it was determined by the responses obtained during the elicitation and the comfort level (of the subjects and the experimenter) with the quality of data. For most subjects we elicited 8 or 9 points (median, 9). Subject 10 required 10 points, and Subject 8 required only 7 points.

With a few noticeable exceptions (see Subjects 1, 4, and 7 in the computerized group), the RMSEs are very low, indicating good fit, especially in the face-to-face group, where the fit is significantly better then in the computerized group \((t(23) = 2.06; p < 0.05)\). However, if the three outliers are eliminated, the goodness of fit in the two groups is comparable.

The median correlation in the face-to-face group was slightly higher than in the computerized experiment \((0.84 \text{ versus } 0.70)\), but the two distributions are not significantly different \((t(23) = −0.39; p > 0.05)\). Given that we are focusing on the copula, we also transformed these correlations to rank-order correlations that do not depend on the marginal distributions. Kruskal (1958) shows that in a bivariate normal distribution, the Spearman \(\rho_s\) and Kendall’s \(\tau_b\) are related to Pearson’s \(\rho\):

\[
\rho_s = \frac{6}{\pi} \sin^{-1}\left(\frac{\rho}{2}\right),
\]

\[
\tau_b = \frac{2}{\pi} \sin^{-1}(\rho).
\]

The results (especially Spearman’s \(\rho_s\)) are very similar. Although the correlations in the face-to-face group appear to be high (Table 2), recall that we also asked the subjects in this group to estimate the correlation at the end of the experiment. Eight of the subjects did (two of them did not provide numbers but simply said that it is positive, which, incidentally, reinforces our point that such correlation judgments are often difficult to provide directly). Among the eight for whom we have direct estimates, the fitted estimate was higher in five cases than the direct estimate, but it was lower for the other three. The medians of the eight correlations were, essentially, identical (median correlation of 0.75 using the isoprobability contour-based approach and 0.73 for the direct estimates). Thus, although it is possible that “objectively” these subjects overestimate the weight–height correlation in the student population; this does not seem to be associated with a particular estimation method.

Note that in four cases (Subject 1 in the face-to-face group and Subjects 3, 11, and 14 in the computerized group), the estimation yielded negative estimates of the correlation coefficient. This counterintuitive result does not reflect an estimation problem. Rather, it can be attributed to the inappropriateness of the bivariate Gaussian assumption for the points assessed for these subjects, as shown in Figure 12. The left panel shows the assessed points for Subject 1 (estimate \(\rho = −0.48\)) and the contours of the various bivariate Gaussian distributions. Clearly, the assessed points are inconsistent with the model. In sharp contrast, the right panel shows one of the best-fitting subjects (Subject 4, estimate \(\rho = 0.92\)), for which many of the points lie exactly on the 0.9 contour.
7.2. Numerical Example: Estimating the Joint Distribution Using a Bivariate Gaussian Copula

Having determined that the isoprobability contour of Subject 4 in the face-to-face group matches a bivariate Gaussian (see Figures 11 and 12), we can estimate the joint cumulative distribution by curve fitting it to a bivariate Gaussian copula. The marginal distribution assessment of weight indicates that \( F_X(88.45) = 0.77 \). Under the assumption of standard Gaussian variable, this implies \( X = 0.74 \). The marginal distribution assessment of height indicates that \( F_Y(177.8) = 0.46 \). This corresponds to \( Y = -0.10 \) in a standard Gaussian distribution. The estimated correlation coefficient was 0.92 (see Table 2). The joint cumulative probability of a standard bivariate Gaussian distribution where \( X = 0.74, Y = -0.10 \), and \( \rho = 0.92 \) is 0.46, which is remarkably close to the previous values obtained by curve fitting to the contour function.

8. Conclusions

We presented a method for direct elicitation and construction of joint probability distributions of continuous variables using isoprobability contours. Our key contribution is the development and empirical validation of a practical, easy to implement assessment method. In addition, we described two methods for constructing the joint probability distribution from the assessed contours and illustrated how to determine bivariate correlations from these values.

The main advantage of the newly proposed approach compared to other approaches such as marginal conditional assessments or direct assessment of dependence is that the judges do not have to provide any numerical responses, or estimate complex joint probabilities (e.g., the quadrant probability) or parameters (e.g., the covariance or correlation between variables) that are not intuitive, prone to judgment errors, and could be intimidating for judges who lack proper statistical training. In our approach, judges only need to express preferences over simple binary gambles with identical payoffs whose probabilities depend on, and reflect, the probability trade-offs between the two variables. A second major advantage of our procedure is that the events being compared involve both variables, so the judges are forced to consider the relevant trade-offs. Thus, the use of isoprobability contours reduces the assessment of joint probability distributions to a one-dimensional cumulative probability assessment (which is cognitively simpler to reason about than conditional assessments) and the elicitation of the probability trade-offs (inferred by expressing preferences over binary deals).

We reported results of a simple feasibility study. The key empirical question we addressed is whether subjects are able to reason about probability trade-offs. The results of the empirical study showed that this is a sensible and reasonable task that was performed successfully. The judgments regarding the joint distributions matched the quality of the standard probability elicitation (FP and FV) for the marginal distributions with respect to difficulty and monotonicity. The monotonicity level was excellent, and the level of internal consistency was satisfactory. Finally, most correlation coefficients estimated from the judges’ assessments were reasonable, and the few outlying results are due to the fact that the judgments of some subjects do not fit the assumptions of the model (in this case, bivariate Gaussian), rather than elicitation or statistical estimation problems. In these types of situations, other models of joint probability distributions may provide better fits.

All these results suggest that people can indeed reason about the newly defined probability trade-off coefficients in assessing the isoprobability contours. It is
important to stress that these results appear to be quite robust. Recall that we obtained comparable results in the computerized group, where about 40% of the series ended before converging to indifference, and we approximated the location of the point on the contour by the middle of the range of values. The one exception to this generalization is that for a minority (3 out of 15) of the subjects in the computerized groups, the fit of the inferred correlations was unsatisfactory.

One of the nice features of the new method is that, unlike the standard methods of assessing dependence, it generates data that can be used in various ways to estimate the target quantities. We discussed and illustrated two ways to estimate the joint probability distribution and infer the bivariate correlation from the judges’ assessments. This flexibility allows us to double check and validate the quality of our procedure. Both methods are easily implemented and generated similar results in our simple example, but in our opinion it is easier to fit contour functions than curve fit bivariate copulas. The former approach requires only the estimation of one probability trade-off and one of the marginal distributions. However, it is also important, as a consistency check, to test that the joint probability density obtained by the contours method is nonnegative, as we do when constructing joint cumulative distributions using cumulative marginal conditional assessments.

The notion of an isoprobability contour leads naturally to several directions for future research. The first is the derivation of different functional forms of contour functions that allow for variable probability trade-offs. For example, all the contour functions we presented and used in our examples involve constant elasticity of probability trade-offs. Other functional forms relaxing this condition, the simplest being a linear function of the variables, i.e., \( \eta(x, y) = \eta_0 + \eta_1 x + \eta_2 y \), which generalizes the forms used in this paper, should be considered.

The second direction pertains to extensions into higher dimensions. The basic principle underlying our approach can be extended beyond the bivariate case, but the complexity of the problem is considerably higher and increases rapidly (as does the complexity of conditioning on a larger number of variables for conditional probability assessments). For example, in the trivariate case, one could estimate relatively easily the probability trade-offs between each pair of variables at fixed values of the third variable. This would lead to conditional probabilities of the form \( F(x_i, x_j | x_k) \) for all distinct permutations of \( i, j, k \), and works perfectly for constructing the joint distribution when the dependence between the variables is of the second order, i.e., when this conditional probability does not depend on \( x_k \). Cases with more complex relationships would require estimation of more complicated three-way probability trade-offs (or, under certain parametric assumptions, repeating the assessment for different values of the variable \( x_k \)), which may be harder to implement and judge. One can also assume a certain functional form of the contour function that extends to higher dimensions and then use the probability trade-offs to estimate its parameters directly. For example, higher-order functions of the form \( C(x, y, z) = xy^z + z^2 \) can also be considered using pairwise trade-offs and their parameters assessed at fixed values of the third variable.

Finally, there are some important empirical directions for future research. The first one is to go beyond the feasibility demonstration of this paper and provide evidence-based guidelines for implementing the new method in practice. This requires answering questions such as the following: How many contours should be assessed? Does it matter which contours are assessed? What is the minimal number of points that should be assessed for each contour, and is there an optimal way of choosing them? It is equally important to compare empirically results based on assessments of isoprobability contours with other direct elicitation methods of dependence. It would be useful to determine which method (or methods) are perceived to be more natural, easier to implement, and more comfortable for (and hence preferred by) the DMs, and verify which method (or methods) generates better (more stable, reliable, accurate, valid, etc.) results in practice.

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**References**


