Valuing Changes in Investment Opportunities

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Arrow and Pratt introduced a measure of risk aversion—the negative ratio of the second to the first derivative of the utility function. This measure has found widespread use in the valuation of uncertain lotteries and in the calculation of the risk premium of an investment. This paper introduces two new measures for characterizing changes in the valuation of uncertain lotteries when their outcomes are modified by a monotone transformation. The first is a characteristic transformation of a utility function, \( U \), and a monotone transformation, \( g \). The shape of the characteristic transformation determines an upper bound, lower bound, or equality on the magnitude of the certainty equivalent of the modified lottery. The second is a measure of change in certainty equivalent, \( \eta \), whose sign also determines upper or lower bounds, and whose magnitude determines the change in value of a “small-risk” lottery when its outcomes are modified by a monotone transformation. For shift (and scale) transformations on the lottery outcomes, both the characteristic transformation and the measure of change, \( \eta \), provide new characterizations for the notions of decreasing absolute (and relative) risk aversion with wealth.

1. Introduction

Von Neumann-Morgenstern’s (1947) expected utility theory asserts that a rational decision maker should value any uncertain lottery, \( X \), by its certainty equivalent, \( CE(X, w) \), defined as the utility inverse of the expected utility,

\[
CE(X, w) = U^{-1} \left( \sum_{i=1}^{n} p_i U(x_i + w) \right) - w, \quad (1)
\]

where \( U \) is the utility function over total wealth, which is assumed to be continuous and monotone; \( x_i \) is an outcome of the lottery, and \( w \) is the initial wealth, which is assumed to be deterministic. If the utility function is strictly concave, then the certainty equivalent of the lottery is less than its mean value, \( \mu \), and so

\[
\mu - CE(X, w) > 0, \quad (2)
\]

in which case the decision maker is said to be risk averse.

Equation (2) shows that the mean value of a lottery is an upper bound on the certainty equivalent if the utility function is concave. Conversely, the mean is a lower bound if the utility function is convex. Finally, the mean is equal to the certainty equivalent if the utility function is linear.

It is natural to think about the magnitude of the difference between the mean value of the lottery and the certainty equivalent (a quantity known as the risk premium). To provide insights into this direction, Pratt (1964) and Arrow (1965) defined a measure of risk aversion, \( \gamma(x) \), as

\[
\gamma(x) = \frac{-U_{xx}(x)}{U_1(x)}, \quad (3)
\]

where \( U_1(x) \) is the first derivative of the utility function and \( U_{xx}(x) \) is the second derivative. The sign of \( \gamma(x) \) determines the decision maker’s attitude towards risk: he is risk averse if the risk-aversion function is positive and is risk seeking if it is negative. The sign of the risk-aversion function therefore determines upper or lower bounds on the certainty equivalent. Pratt also showed that the risk premium of a lottery can be approximated in terms of the magnitude of the risk aversion function (for small risks) using the quantity \( (\sigma^2/2)\gamma(\mu) \), where \( \sigma^2 \) is the variance of the lottery.

Our focus in this paper is the change in value of a lottery when its outcomes are modified by a monotone transformation. For example, we might be interested in the value gained due to cost savings in an investment project, which corresponds to a shift transformation on the lottery outcomes. Alternatively, cost savings might result in some project delay corresponding to a discount factor, which results in a linear (or even nonlinear) transformation on the lottery outcomes depending on the cost structure of the firm. It would be convenient to have a method that characterizes the change in the certainty equivalent in this case.

This paper introduces two new measures for characterizing changes in investment opportunities. The first is a characteristic transformation of a utility function, \( U \), and a monotone transformation, \( g \). The shape of the characteristic transformation determines an upper bound, lower bound, or equality on the certainty equivalent of a lottery.
when its outcomes are modified by a monotone transformation. The second is a measure of change in certainty equivalent, \( \eta_i \), whose sign determines whether the certainty equivalent of the modified lottery is above or below some reference value. The reference value is chosen to be the same transformation, \( g \), applied to the certainty equivalent of the unmodified lottery. For small risks, the magnitude of \( \eta_i \) plays an important role in quantifying the change in certainty equivalent, and it also relates the certainty equivalent of the modified lottery to the mean, variance, and certainty equivalent of the unmodified lottery.

The characteristic transformation and the measure, \( \eta_i \), provide new characterizations (and generalizations) for the notions of decreasing absolute (and relative) risk aversion with wealth. To illustrate, Pratt interpreted decreasing risk aversion as the condition that the risk premium of a lottery is a decreasing function of wealth. Yaari (1969) and Dybvig and Lippman (1983) provided an equivalent appealing interpretation of decreasing risk aversion: that a gamble accepted at one wealth level will be accepted at all higher levels, i.e.,

\[ CE(X, w + z) - CE(X, w) > 0, \quad z > 0. \]  

(4)

Abbas and Aczél (2010) provided a related interpretation of decreasing risk aversion: that the certainty equivalent of an uncertain lottery increases by an amount that is greater than \( z \) when all the lottery outcomes increase by \( z \), i.e.,

\[ CE(X + z, w) - CE(X, w) > z, \quad z > 0, \]  

(5)

where \( X + z \) represents a lottery whose outcomes are modified by a fixed amount \( z \). Thus, the assertion of decreasing risk aversion with wealth implies that the magnitude of the certainty equivalent of the modified lottery, \( CE(X + z, w) \), is higher than some reference value, \( CE(X, w) + z \). In our work we show that, for shift transformations on the lottery outcomes, both the sign of \( \eta_i \) and the shape of the characteristic transformation determine the relative magnitudes of \( CE(X + z, w) \) and \( CE(X, w + z) \), and thereby determine decreasing, increasing, or constant absolute risk aversion with wealth. For more general transformations, the new measures define a similar relationship between the modified certainty equivalent and some reference value.

The remainder of this paper is organized as follows. Section 2 introduces a characteristic transformation of a utility function, \( U \), and a monotone transformation, \( g \). Section 3 introduces a measure of change in certainty equivalent whose sign determines upper or lower bounds on the new certainty equivalent. Section 4 interprets the magnitude of this new measure for small risks. Section 5 presents conclusions and directions for future work.

2. The Characteristic Transformation of a Utility Function and a Monotone Transformation

2.1. Upper and Lower Bounds on the New Certainty Equivalent

Our focus will be continuous and strictly monotone utility functions, \( U \), and continuous and strictly increasing transformations, \( g \). For convenience, we define utility functions and lotteries over total wealth, \( y \), where \( y = x + w \). We begin our development with the following problem:

Suppose you face an uncertain investment whose prospects are characterized by the lotteries \( \langle p_1, y_1; \ldots; p_n, y_n \rangle \), and you value this lottery by a wealth equivalent, \( \tilde{y} \), where \( \tilde{y} = U^{-1}(\sum_{i=1}^{n} p_i U(y_i)) \). Now suppose that due to some changes in this investment project, all the lottery outcomes are modified by a monotone transformation, \( g \), to yield the modified lottery \( \langle p_1, g(y_1); \ldots; p_n, g(y_n) \rangle \). How does the certainty equivalent of the modified lottery, which we denote as \( \tilde{g} \), compare to \( g(\tilde{y}) \)?

It is straightforward to see that if the lottery is (i) deterministic, or (ii) if \( g(y) = y \), then \( \tilde{g} = g(\tilde{y}) \). Our main focus will therefore be uncertain lotteries and transformations \( g(y) \neq y \). We consider three scenarios:

(i) For a given utility function, if the transformation is of the form \( g(y) = U^{-1}(kU(y) + d) \) (Abbas 2007). This result has several implications:

- (i) For a given utility function, if the transformation is of the form \( g(y) = U^{-1}(kU(y) + d) \), then we can calculate the certainty equivalent of the modified lottery by simply applying the same monotone transformation \( g \) to the previously calculated certainty equivalent. This observation simplifies the calculation of the new certainty equivalent significantly.

- (ii) Conversely, if we can assert that \( \tilde{g} = g(\tilde{y}) \) for all uncertain lotteries, then we can derive the functional form of the utility function by solving the functional equation \( U(g(y)) = kU(y) + d \). A special case applies to shift transformations, \( g(y) = y + z \); i.e., if we can assert that the certain equivalent of the lottery \( \langle p_1, y_1 + z; \ldots; p_n, y_n + z \rangle \) is equal to \( \tilde{y} + z \) for values of \( z \) on an interval of positive length, then the utility function must satisfy the equation

\[ U(y + z) = k(z)U(y) + d(z) \]  

for values of \( z \) on an interval. This condition implies that the utility function is either linear or exponential (see, for example, Pfanzagl 1959, Howard 1967, Raiffa 1968, and Abbas and Aczél 2010).

(ii) If \( \tilde{g} > g(\tilde{y}) \) for all uncertain lotteries, then \( g(\tilde{y}) \), which is straightforward to calculate, provides a lower bound on the new certainty equivalent. To calculate this lower bound, we simply apply a monotone transformation to the previously calculated certainty equivalent.
(3) If \( \hat{g} < g(\hat{y}) \) for all uncertain lotteries, then \( g(\hat{y}) \) provides an upper bound on the new certainty equivalent. Once again, this upper bound is straightforward to calculate.

But how can we assert whether \( g(\hat{y}) \) is an upper bound, lower bound, or equality on \( \hat{g} \)? And can we quantify \( \hat{g} \) when \( \hat{g} \neq g(\hat{y}) \)?

**Definition (The Characteristic Transformation of \( U \) and \( g \)).** The characteristic transformation, \( m \), of a utility function, \( U \), and a transformation, \( g \), is defined by

\[
m(t) \equiv U(g(U^{-1}(t))).
\]

(6)

The following theorem illustrates the implications of this definition.

**Theorem 1 (Bounds on the Changes in Investment Opportunities).** Given a utility function, \( U \), and a transformation, \( g \), applied to the lottery outcomes, the following statements hold:

(a) \( \hat{g} = g(\hat{y}) \) for all lotteries if and only if \( m(t) \) is a linear function of \( t \).

(b) \( \hat{g} > g(\hat{y}) \) for all lotteries if and only if \( m(t) \) is a strictly convex function of \( t \).

(c) \( \hat{g} < g(\hat{y}) \) for all lotteries if and only if \( m(t) \) is a strictly concave function of \( t \).

Theorem 1 explains the importance of the characteristic transformation: its shape determines whether we have an upper bound, lower bound, or equality on the value of the new certainty equivalent in relation to the reference value \( g(\hat{y}) \).

Theorem 1 also shows that the deviation of the characteristic transformation from linearity drives the inequality between \( \hat{g} \) and \( g(\hat{y}) \). If the characteristic transformation is linear, then \( \hat{g} = g(\hat{y}) \). As the characteristic transformation deviates from linearity, being either strictly concave or convex, \( \hat{g} \) deviates from \( g(\hat{y}) \). Theorem 1 enables us to determine the bounds on the new certainty equivalent in this case. The following examples illustrate some applications of this result.

**Example 1 (Modifying Lotteries by Shift Transformations).** Suppose that, due to cost savings, the outcomes of a lottery are modified by a shift transformation

\[
\hat{g}(t) = t + \hat{z}, \quad \hat{z} > 0.
\]

By definition, the characteristic transformation is

\[
m(t) = U(U^{-1}(t) + \hat{z}), \quad \hat{z} > 0,
\]

for all values of \( t \) that lie on the range of the utility function. We now calculate \( m(t) \) for various utility functions to determine the relation between \( \hat{g} \) and \( g(\hat{y}) \) in this case.

1. For linear and exponential utility functions: Linear: \( U(y) = cy + w \Rightarrow m(t) = t + cz \), and Exponential: \( U(y) = a + be^{-y} \Rightarrow m(t) = te^{-z} + a(1 - e^{-z}) \), both of which are linear in \( t \). From Theorem 1, we can assert the equality

\[
\hat{g} = g(\hat{y}) = \hat{y} + \hat{z}
\]

with these utility functions. Theorem 1 thus provides a simple method to verify Pfanzagl’s (1959) linear and exponential result.

2. For a logarithmic utility function,

\[
U(y) = \ln(y) \Rightarrow m(t) = \ln(e^t + \hat{z}),
\]

which is a strictly convex function of \( t \) when \( \hat{z} > 0 \). From Theorem 1, we can assert that

\[
\hat{g} > \hat{y} + \hat{z}, \quad \hat{z} > 0,
\]

and therefore \( \hat{y} + \hat{z} \) presents a lower bound on the new certainty equivalent.

3. For a power utility function,

\[
U(x) = (x + w)^\alpha \Rightarrow m(t) = (t^{1/\alpha} + \hat{z})^\alpha, \quad \hat{z} > 0,
\]

which is a strictly concave function of \( t \) when \( \alpha > 1 \), asserting that

\[
\hat{g} < \hat{y} + \hat{z}
\]

is a strictly convex function of \( t \) when \( \alpha < 1 \), asserting that

\[
\hat{g} > \hat{y} + \hat{z},
\]

and is a linear function of \( t \) when \( \alpha = 1 \), asserting that

\[
\hat{g} = \hat{y} + \hat{z}.
\]

Thus, we can get either an upper, lower bound or equality depending on the value of \( \alpha \).

Note that the characteristic transformation does not depend on the initial wealth, \( w \), for shift transformations, i.e., for shift transformations on the lottery outcomes, both \( U(x + w_i) \) and \( U(x + w_j) \) have the same characteristic transformation.

### 2.2. Characterizing Increasing and Decreasing Risk Aversion

#### 2.2.1. Characterizing Decreasing Absolute Risk Aversion

The characteristic transformation can also be used to characterize increasing (and decreasing) risk aversion with wealth. To illustrate, consider the shift transformation \( g(t) = t + \hat{z}, \hat{z} > 0 \) and its characteristic transformation \( U(U^{-1}(t) + \hat{z}), \hat{z} > 0 \).

**Proposition 1 (Characterizing Absolute Risk Aversion Using Shift Transformations).** Given a twice-differentiable utility function, \( U \), with positive first derivative, and a characteristic transformation, \( m(t) = U(U^{-1}(t) + \hat{z}), \forall \hat{z} > 0 \). The following conditions hold

(a) \( U \) exhibits constant absolute risk aversion if and only if \( m(t) \) is a linear function of \( t \).
(b) $U$ exhibits decreasing absolute risk aversion if and only if $m(t)$ is a convex function of $t$.
(c) $U$ exhibits increasing absolute risk aversion if and only if $m(t)$ is a concave function of $t$.

Proposition 1 enables us to characterize absolute risk aversion for a given utility function using the characteristic transformation without actually calculating the risk aversion function itself. For example, the characteristic transformation of an exponential utility function is $m(t) = te^{c_t} + a(1 - e^{-c_t})$, which is a linear function of $t$, and so we can assert constant absolute risk aversion. Proposition 1 also enables us to assert decreasing absolute risk aversion. To illustrate, recall that for a logarithmic utility function and a shift transformation, $m(t) = \ln(e^t + z)$, which is a convex function of $t$ when $z > 0$. From Proposition 1, we can now assert that this utility function exhibits decreasing absolute risk aversion without actually calculating the risk aversion function itself. In the proof of Proposition 1, we also show that the concavity/convexity results of Proposition 1 are reversed when $g(t) = t + z, z < 0$.

2.2.2. Characterizing Decreasing Relative Risk Aversion. The characteristic transformation can also be used to characterize the relative risk-aversion function, $r(y) = y^2(y)$, using scale transformations, $g(t) = zt, z > 1$, which corresponds to a characteristic transformation $m(t) = U(zU^{-1}(t)), z > 1$, as we illustrate below.

Proposition 2 (Characterizing Decreasing Relative Risk Aversion with Scale Transformations). Given a twice-differentiable utility function, $U$, with positive first derivative, and a characteristic transformation $m(t) = U(zU^{-1}(t)), z > 1$, the following conditions hold.
(a) $U$ exhibits constant relative risk aversion if and only if $m(t)$ is a linear function of $t$.
(b) $U$ exhibits decreasing relative risk aversion if and only if $m(t)$ is a convex function of $t$.
(c) $U$ exhibits increasing relative risk aversion if and only if $m(t)$ is a concave function of $t$.

The following examples illustrate an application of this result:

(1) For a logarithmic utility function, $U(y) = \ln(y)$, $m(t) = U(zU^{-1}(t)) = \ln(ze^t) = t + \ln z$, which is a linear function of $t$, and so we can assert that it exhibits constant relative risk aversion without actually calculating the risk aversion function itself.

(2) For an increasing exponential utility function, $U(y) = -e^{-\gamma y}, \gamma > 0$, $m(t) = -(1 - e^{-t}), -1 \leq t \leq 0$, which is a concave function of $t$ when $z > 1$, and so it exhibits increasing relative risk aversion. As we show in the proof of Proposition 2, the concavity/convexity results of Proposition 2 are also reversed when $g(t) = zt, 0 < z < 1$.

Note that the characteristic transformation is invariant to a scale transformation in this case, i.e., both $U(c,y)$ and $U(c,y)$ have the same characteristic transformation for scale transformations.

2.3. The Effects of a Linear Transformation on the Utility Function

If both $U$ and $g$ are strictly increasing functions, then it is straightforward to see that $m(t)$ is also strictly increasing. We now consider the effects that a linear transformation on the utility function has on its characteristic transformation.

Proposition 3. If $m(t)$ is the characteristic transformation of $U, g$, then the characteristic transformation of $a + bU, g$ is $a + bm(t - a)/b$.

Proposition 3 has several implications. First, it asserts that the shape of $m(t)$ does not change by applying a shift transformation, $a$, to the utility function because it merely results in a positive increment of $a$ in the characteristic transformation. Second, it asserts that if $g$ is strictly increasing, then $m(t)$ is a strictly increasing function for any strictly monotone utility function (increasing or decreasing). To illustrate this fact, note that if $m(t)$ is strictly increasing, then $-m(-t)$, the transformation of $-U, g$, is also strictly increasing. Third, Proposition 3 asserts that applying a positive scale amount, $b$, to the utility function has no effect on the concavity/convexity/linearity of $m(t)$.

On the other hand, applying a negative scale transformation to the utility function reverses the concavity/convexity of $m(t)$. To illustrate, note that if $m(t)$ is a strictly increasing concave function, then $-m(-t)$ is a strictly increasing convex function. Therefore, the concavity/convexity results of Theorem 1 (and Propositions 2 and 3) would need to be reversed for decreasing utility functions. In the next section, we present a measure that takes this scaling effect into account and is therefore invariant to a linear transformation on the utility function.

3. A Measure of Change in Certainty Equivalent

Our focus will now be utility functions and transformations that are twice continuously differentiable. We assume that the first derivative of the utility function is either positive or negative, whereas the first derivative of the monotone transformation is positive.

Define the measure $\eta(y)$ as

$$\eta(y) = \frac{g(y)\gamma(g(y)) - \gamma(y) - \frac{g(y)}{g(y)}\gamma'}{g(y)}$$

(7)

where $g(y), g, \gamma$ are the first and second derivatives of $g(y)$ with respect to $y$, respectively. If we further define the composite transformation $U^{\text{Mod}}(y) = U(g(y))$ and its risk-aversion function, $\gamma^{\text{Mod}}(y) = \gamma'(g(y))/U^{\text{Mod}}(y)$, then, as we show in the appendix, we have an equivalent definition for $\eta(y)$ as

$$\eta(y) = \frac{\gamma^{\text{Mod}}(y) - \gamma(y)}{\gamma^{\text{Mod}}(y)}$$

This expression provides both a compact representation of $\eta(y)$ and an interpretation for its magnitude in terms of
the difference between two risk aversion functions, $\gamma^{mod}(y)$ and $\gamma(y)$. The following theorem presents an application of the sign of $\eta_y(y)$.

**Theorem 2.** The measure $\eta_y(y)$ is invariant under a linear transformation on the utility function, and it satisfies the following conditions

$$\text{sgn}(\eta_y(y)) = \begin{cases} 
+1 & \forall y \ni \tilde{g} < g(\tilde{y}) \text{ for all lotteries,} \\
0 & \forall y \ni \tilde{g} = g(\tilde{y}) \text{ for all lotteries,} \\
-1 & \forall y \ni \tilde{g} > g(\tilde{y}) \text{ for all lotteries.}
\end{cases}$$

Theorem 2 shows that the sign of $\eta_y(y)$ determines the relative magnitude of $\tilde{g}$ in relation to the reference value, $g(\tilde{y})$, and therefore provides bounds on the new certainty equivalent. This result is invariant to a linear transformation on the utility function. From Theorems 1 and 2, we can also see how the sign of $\eta_y(y)$ and the shape of the characteristic transformation (linear/strictly concave/strictly convex) are related.

The following examples illustrate an application of Theorem 2.

1. **Shift Transformations:**

$$g(y) = y + z, \quad z > 0 \implies g_y(y) = 1, \quad g_{yy}(y) = 0.$$  

By definition,

$$\eta_y(y) = \gamma(y + z) - \gamma(y).$$

Equation (8) shows that for a shift transformation, $\eta_y(y)$ is zero when $\gamma(y + z) = \gamma(y)$ for all values of $y$ and $z > 0$. This implies constant absolute risk aversion. The measure $\eta_y(y)$ is negative under shift transformations when $\gamma(y + z) < \gamma(y)$, $z > 0$, which implies (strictly) decreasing absolute aversion. Finally, $\eta_y(y)$ is positive under shift transformations when $\gamma(y + z) > \gamma(y)$, $z > 0$, which implies strictly increasing absolute aversion. Thus, the sign of $\eta_y(y)$ is by itself an indication of increasing, decreasing, and constant absolute risk aversion for shift transformations. It is straightforward to see that these results are reversed for $z < 0$.

Theorem 2 further relates the sign of $\eta_y(y)$ to the difference between $\tilde{g}$ and $\tilde{y} + z, z > 0$:

- (i) If $\eta_y(y) > 0$ (increasing absolute risk aversion),
  then $\tilde{g} < \tilde{y} + z$;

- (ii) if $\eta_y(y) = 0$ (constant absolute risk aversion),
  then $\tilde{g} = \tilde{y} + z$. This is Pflanzagl’s (1959) condition for linear and exponential utility functions. Finally,

- (iii) if $\eta_y(y) < 0$ (decreasing absolute risk aversion),
  then $\tilde{g} > \tilde{y} + z$.

In §4, we show how the magnitude of $\eta_y(y)$ characterizes the inequality gap between $\tilde{g}$ and $\tilde{y} + z$ for small risks.

2. **Scale Transformations:**

$$g(y) = zy, \quad z > 1 \implies g_y(y) = z, \quad g_{yy}(y) = 0.$$  

By definition,

$$\eta_y(y) = z\gamma(zy) - \gamma(y) = \frac{r(zy) - r(y)}{y},$$

where $r(y) = y\gamma(y)$ is the relative risk-aversion function. For scale transformations, $\eta_y(y)$ is zero when $r(zy) = r(y)$, for $y > 0$, $z > 1$, which implies constant relative risk aversion. The measure $\eta_y(y)$ is negative when $r(zy) < r(y)$, $z > 1$, which implies (strictly) decreasing relative risk aversion with wealth. Finally, $\eta_y(y)$ is positive when $r(zy) > r(y)$, $z > 1$, which implies (strictly) increasing relative risk aversion with wealth. Thus, the sign of $\eta_y(y)$ is an indication of increasing, decreasing, and constant relative risk aversion for scale transformations. It is straightforward to see that these results are reversed for $0 < z < 1$.

Theorem 2 also implies that

- (i) if $\eta_y(y) > 0$ (increasing relative risk aversion),
  then $\tilde{g} < \tilde{y}z \text{ or } \tilde{g}/\tilde{y} < z$;

- (ii) if $\eta_y(y) = 0$ (constant relative risk aversion),
  then $\tilde{g} = \tilde{y}z$. Finally,

- (iii) if $\eta_y(y) < 0$ (decreasing relative risk aversion),
  then $\tilde{g} > \tilde{y}z \text{ or } \tilde{g}/\tilde{y} > z$.

In §4, we show how the magnitude of $\eta_y(y)$ characterizes the inequality gap between $\tilde{g}$ and $\tilde{y}z$ for small risks.

3. **Nonlinear Monotone Power Transformations:** Linear transformations can be interpreted in terms of receiving a fraction of the lottery (or a delayed lottery) and a receiving a deterministic monetary amount in return. For a linear transformation,

$$g(y) = z_1y + z_2, \quad z_1 > 1, z_2 > 0$$

$$\implies g_y(y) = z_1 \quad \text{and} \quad g_{yy}(y) = 0.$$  

By definition,

$$\eta_y(y) = z_1\gamma(z_1y + z_2) - \gamma(y).$$

For linear transformations, $\eta_y(y)$ is zero when $z_1\gamma(z_1y + z_2) = \gamma(y)$, implying $\tilde{g} = z_1\tilde{y} + z_2$.

It is negative when $z_1\gamma(z_1y + z_2) < \gamma(y)$, which implies $\tilde{g} > z_1\tilde{y} + z_2$. Finally, $\eta_y(y)$ is positive when $z_1\gamma(z_1y + z_2) > \gamma(y)$, which implies $\tilde{g} < z_1\tilde{y} + z_2$.

4. **Nonlinear Monotone Power Transformations:** So far, we have considered transformations where the second derivative, $g_{yy}(y)$, is zero. For more general monotone transformations, this need not be the case. For power transformations with odd exponents, as an example, we get,

$$g(y) = y^z \implies g_y(y) = zy^{z-1}, \quad g_{yy}(y) = (z-1)y^{z-2}.$$  

By definition,

$$\eta_y(y) = zy^{z-1}\gamma(y^z) - \gamma(y) - \frac{z-1}{y},$$

from which we can tell the relative magnitude of change in certainty equivalent for a given utility function.
4. Valuing Changes in the Small

As we discussed, if the lottery is deterministic, or if \( g(y) = y \), then \( ˜g = g(\bar{y}) \). For more general lotteries and transformations, however, this equality need not hold. Our focus in this section will be the asymptotic behavior of the change in certainty equivalent for small-risks: uncertain lotteries whose support, \([y_{\min}, y_{\max}]\), approaches a local neighborhood around the mean, \( \mu \). We examine the change in certainty equivalent as the difference \( (y_{\max} - y_{\min}) \to 0 \).

As we shall see, the magnitude of \( \eta \), plays an important role in quantifying the difference between \( g(\bar{y}) \) and \( ˜g(\bar{y}) \) in this setting. The following theorem characterizes the change in certainty equivalent for small risks. The “small \( \sigma' \)” notation that is used in this theorem implies that \( \lim_{\delta \to 0}(\sigma(\delta))/\delta = 0 \).

**Theorem 3 (Valuing Changes for Small Risks).**

\[
\tilde{g} = g\left(\bar{y} - \frac{\sigma^2}{2} \eta_y(\mu) + o(\sigma^2)\right),
\]

as \( (y_{\max} - y_{\min}) \to 0 \). \( (12) \)

Theorem 3 interprets the magnitude of \( \eta_y \) in terms of the change in certainty equivalent for small-risk lotteries whose domain forms a local neighborhood around the mean. It asserts that the term \( (\sigma^2/2)\eta_y(\mu) \) fills in the inequality gap up to an order of \( o(\sigma^2) \) in the argument of \( g \). In the limit, when the lottery is deterministic, the support becomes the mean, and so \( \sigma' = 0 \). From Theorem 3, this implies that \( \tilde{g} = g(\bar{y}) \), as expected. The following examples illustrate an application of Theorem 3.

**Example 2 (Valuing Small Changes Due to Shift Transformations).** Consider a shift transformation \( g(y) = y + z \), that may be the result of cost savings in an investment project. From (8), \( \eta_y(\mu) = \gamma(\mu + z) - \gamma(\mu) \), and from Theorem 3,

\[
\tilde{g} = \bar{y} + z - \frac{\sigma^2}{2} \eta_y(\mu) + o(\sigma^2),
\]

as \( (y_{\max} - y_{\min}) \to 0 \). \( (13) \)

As the support of the lottery approaches zero, the new certainty equivalent due to shift transformations can be approximated, up to an order of \( o(\sigma^2) \), by

\[
\tilde{g} \approx \bar{y} + z - \frac{\sigma^2}{2} \eta_y(\mu), \quad \text{as} \quad (y_{\max} - y_{\min}) \to 0.
\]

It is clear from the approximate expression that if \( \eta_y \) is positive, then \( \tilde{g} < \bar{y} + z \), and if it is negative, then \( \tilde{g} > \bar{y} + z \).

**Example 3 (Valuing Small Changes Due to Scale Transformations).** Consider a scale transformation, \( g(y) = zy, z > 1, y > 0 \). From (9), \( \eta_y(y) = z\gamma(zy) - \gamma(y) \), and from Theorem 3,

\[
\tilde{g} = z\left(\bar{y} - \frac{\sigma^2}{2} \eta_y(\mu) + o(\sigma^2)\right), \quad \text{as} \quad (y_{\max} - y_{\min}) \to 0.
\]

Alternatively, we may write

\[
\tilde{g} = z\left(1 - \frac{\sigma^2}{2} \eta_y(\mu) + o(\sigma^2)\right), \quad \text{as} \quad (y_{\max} - y_{\min}) \to 0,
\]

which we denote as the fractional change in certainty equivalent. For small risks, we may write as an approximation, up to the order of \( o(\sigma^2) \),

\[
\tilde{g} \approx z\left(1 - \frac{\sigma^2}{2} \eta_y(\mu)\right), \quad \text{as} \quad (y_{\max} - y_{\min}) \to 0. \quad (14)
\]

**Example 4 (Characterizing Jensen’s Inequality Gap).** As a special case of Theorem 3, consider a linear utility function whose certainty equivalent is equal to the expected value of the lottery. By direct substitution for a linear utility function, the measure of change is

\[
\eta_y(y) = -\frac{g_{yy}(y)}{g_y(y)},
\]

and from Theorem 3, with \( \tilde{y} = \mu, \tilde{g} = E[g(y)] \), we get

\[
E[g(y)] = g\left(\mu + \frac{\sigma^2}{2} \eta_y(\mu) + o(\sigma^2)\right),
\]

as \( (y_{\max} - y_{\min}) \to 0 \). \( (15) \)

Equation (15) fills in Jensen’s inequality gap for small risks. If \( g \) is strictly concave, then \( g_{yy} < 0 \), and so \( E[g(y)] < g(\mu) \), an indication of Jensen’s inequality. On the other hand, if \( g \) is strictly convex, then \( g_{yy} > 0 \), and so \( E[g(y)] > g(\mu) \). The inclusion of the term \( (\sigma^2/2)(g_{yy}(\mu)/g_y(\mu)) \) on the right-hand side fills in the Jensen’s inequality gap for nonlinear transformations and small risks.

The following example presents another application of the magnitude of \( \eta_y \) for interpreting the parameters of a utility function.

**Example 5 (Interpreting the Parameters of the Functional Form).** Suppose that a person with an exponential utility function has a risk-aversion coefficient, \( \gamma \). We can interpret the effects that the magnitude of the risk-aversion coefficient has on changes in the certainty equivalent for small risks. Consider, for example, a scale transformation, \( g(y) = zy \). From (12), we write as an approximation for small risks,

\[
\tilde{g} \approx g\left(z\left(\bar{y} - \frac{\sigma^2}{2} \eta_y(\mu)\right)\right) = z\left(\bar{y} - \frac{\sigma^2}{2} \eta_y(\mu)\right). \quad (16)
\]

From (9), the value of \( \eta_y \) for a scale transformation is

\[
\eta_y(y) = z\gamma(zy) - \gamma(y).
\]

For an exponential utility function, this implies \( \eta_y(y) = (z - 1)\gamma \). Substituting into (16) and rearranging gives

\[
\gamma \approx \frac{2(\bar{y} - \tilde{g})}{\sigma^2 z(\bar{z} - 1)}. \quad (17)
\]
5. Conclusion

Risk aversion is concerned with the valuation of investment opportunities. The negative ratio of the second to the first derivative of the utility function is a measure of risk aversion that helps quantify the risk premium of an investment. In a dynamic world, changes may occur in our investment portfolios. Change may be the result of an updated forecast or due to a proactive intervention such as a decision to share the risk with other partners.

We introduced the notion of a characteristic transformation for a utility function \( U \) and a transformation \( g \) to help the analyst characterize changes in investment opportunities when their outcomes are modified by monotone transformations. The concavity (convexity) of the characteristic transformation provides an upper (lower) bound on the new certainty equivalent. Applying these results in the reverse direction (and asserting the relative magnitude of the change) is also helpful in characterizing certain properties of the utility function. For example, knowing whether the certainty equivalent is above or below some reference value determines the shape of the characteristic transformation (concave or convex). This latter result is particularly useful for shift and scale transformations on the lottery outcomes because it asserts either decreasing or increasing absolute (and relative) risk aversion with wealth.

We also introduced a measure of change in certainty equivalent that is invariant to a linear transformation on the utility function. The sign of this measure provides upper and lower bounds on the new certainty equivalent for monotone changes in the lottery outcomes. For small risks, the magnitude of \( \eta \) also characterizes the actual magnitude of the change in certainty equivalent, and it relates the mean, variance, and certainty equivalent of the unmodified lottery.

The characteristic transformation and the measure of change in certainty equivalent augment previous work on characterizing the properties of utility functions. For example, it has been known since Pfanzagl (1959) that linear and exponential utility functions uniquely satisfy the delta property (i.e., when all outcomes of a lottery increase by a fixed amount \( z \), the certainty equivalent increases by \( z \)). This paper augments this work in several directions. First, it provides upper and lower bounds on the new certainty equivalent when the utility function is not linear or exponential. Second, it quantifies the actual magnitude of the deviation of the new certainty equivalent for small risks. Third, the results apply to more general monotone transformations than just shift transformations on the lottery outcomes.

The measure of change in certainty equivalent leads to several directions for future research. From a theoretical perspective, future work can focus on strong measures of change (along the lines of strong risk aversion, Ross 1981), that guarantee that one utility function will lead to a higher change in certainty equivalent than another when a lottery is modified by a monotone transformation. Future work can also derive measures of change in certainty equivalent for multivariate lotteries that build on the notion of multivariate risk aversion (Richard 1975), as well as further expansions to include third (and higher-order) moments. Finally, future work can provide empirical investigations into the ease and comfort of asserting relative changes in certainty equivalents in relation to some reference value.

Appendix

Throughout the proofs we use a summation sign for discrete lotteries. The results for continuous lotteries having continuous cumulative distributions are identical, replacing the summation sign with an integral. The proofs for the continuous case have therefore been omitted.

Proof of Theorem 1. Sufficiency: If \( m(t) = U(g(U^{-1}(t))) \) is a strictly convex function of \( t \), then by Jensen’s inequality, \( m\left(\sum_{i=1}^{n} p_i t_i\right) < \sum_{i=1}^{n} p_i m(t_i) \) for \( \sum_{i=1}^{n} p_i = 1, p_i > 0 \). Put \( t_i = U(y_i) \). This implies that

\[
U\left(g\left(U^{-1}\left(\sum_{i=1}^{n} p_i U(y_i)\right)\right)\right) < \sum_{i=1}^{n} p_i U(g(y_i)).
\]

Rearranging gives

\[
g\left[U^{-1}\left(\sum_{i=1}^{n} p_i U(y_i)\right)\right] < U^{-1}\left(\sum_{i=1}^{n} p_i U(g(y_i))\right),
\]

or \( g(y) < \tilde{g} \). The proof of strict concavity follows the exact same analysis by reversing the inequality sign, and the proof of linearity follows from Jensen’s equality instead of the inequality.

Necessity: If \( g(y) < \tilde{g} \), then \( g[U^{-1}(\sum_{i=1}^{n} p_i U(y_i))] < U^{-1}(\sum_{i=1}^{n} p_i U(g(y_i))). \)

Define \( V_g(y) = U(g(y)) \), and substitute to get

\[
V_g^{-1}\left(\sum_{i=1}^{n} p_i V_g(y_i)\right) > U^{-1}\left(\sum_{i=1}^{n} p_i U(y_i)\right).
\]
Define $y_i = U^{-1}(t_i)$, $f = V_i(U^{-1})$, and substitute to get $(\sum p_i f(t_i)) > \sum p_i f(t_i)$, which implies that $f$ is a strictly convex function. Hence, $U(g(U^{-1}(y))) = m(y)$ is a strictly convex function of $y$. The proof of strict concavity follows by reversing the inequality sign, and the proof of linearity follows from Jensen’s inequality.

**Proof of Proposition 1.** $\Rightarrow$: If $m(t) = U(U^{-1}(t) + z)$, then $m(U(y)) = U(y + z)$. Taking the first derivative gives $U_y(y + z) = m_y(U(y))U_y(y)$. Because $U_y > 0$, this implies that $m_y > 0$. Taking the second derivative gives $U_{yy}(y + z) = m_{yy}(U(y))(U_y(y))^2 + m_y(U_y(y))U_{yy}(y)$.

Because $U_y > 0$, we can divide the second derivative by the first to get

$$\gamma(y + z) = U_y(U(y))m_{yy}(U(y))m_y(U_y(y)) + \gamma(y).$$

Rearranging gives

$$\gamma(y + z) - \gamma(y) = -U_y(U(y))\frac{m_{yy}(U(y))}{m_y(U_y(y))}.$$

(18)

Because $U_y(y), m_y(y) > 0$, the sign of $\gamma(y + z) - \gamma(y)$ is determined by the sign of $m_{yy}$. Hence, $\gamma(y + z) - \gamma(y) \leq 0$ if $m(t)$ is a convex function. This also implies that $\gamma(y + z) \leq \gamma(y)$. For $z > 0$, this implies decreasing risk aversion with wealth, and for $z < 0$, this implies increasing risk aversion.

$\Leftarrow$: If $U$ exhibits decreasing risk aversion, then $\gamma(y + z) - \gamma(y) \leq 0$, $z > 0$. From (18), this implies that

$$-U_y(U(y))\frac{m_{yy}(U(y))}{m_y(U_y(y))} \leq 0,$$

This implies that $m_{yy}(t) \geq 0$, and so $m(t)$ is convex.

The proof of increasing risk aversion follows the same analysis, reversing the inequality in Jensen’s inequality. The proof of constant absolute risk aversion replaces the inequality with equality.

**Proof of Proposition 2.** $\Rightarrow$: If $m(t) = U(U^{-1}(t))$, then $m(U(y)) = U(y)$ is a convex function. Taking the first and second derivatives gives

$$zU_y(z) = m_y(U(y))U_y(y) \quad \text{and} \quad z^2U_{yy}(z) = m_{yy}(U(y))(U_y(y))^2 + m_y(U_y(y))U_{yy}(y).$$

Hence, $z\gamma(y) = -U_y(U(y))(m_{yy}(U(y)))/(m_y(U_y(y))) + \gamma(y)$. Multiplying both sides by $y$ and rearranging gives

$$z\gamma(y) - y\gamma(y) = -yU_y(U(y))\frac{m_{yy}(U(y))}{m_y(U_y(y))}.$$

(19)

which is nonpositive if $m(t)$ is a convex function. This is implies that

$$r(y_1) - r(y) = -yU_y(U(y))\frac{m_{yy}(U(y))}{m_y(U_y(y))} \leq 0,$$

where $r(y) = y\gamma(y)$ is the relative risk aversion function and $\gamma_i = zy$.

For $z > 1$, this implies decreasing relative risk aversion, and for $0 < z < 1$, this implies increasing relative risk aversion.

$\Leftarrow$: If $U$ exhibits decreasing relative risk aversion, then $r(z) - r(y) \leq 0$, $z > 1$. From (19), this implies that $-yU_y(U(y))(m_{yy}(U(y)))/(m_y(U_y(y))) \leq 0$. If both $U_y(y) > 0$, then $m_y(y) > 0$, and this implies that $m_{yy}(t) \geq 0$, and so $m(t)$ is convex.

The proof of increasing risk aversion follows the same analysis reversing the inequality in Jensen’s inequality. The proof of constant relative risk aversion replaces the inequality with equality.

**Proof of Proposition 3.** If the characteristic transformation of $U, g$ is $m(t)$, then by definition, the characteristic transformation of $U_l(x) = a + bU(x)$ is

$$U_l(g(U^{-1}(t))) = a + bU \left( g \left( U^{-1} \left( t - \frac{a}{b} \right) \right) \right)$$

$$= a + bm \left( t - \frac{a}{b} \right).$$

**Equivalent definition of the measure $\eta$.** Define $\eta \equiv U^{\text{mod}}(y) = U(g(y))$, and $\gamma \equiv U^{\text{mod}}(y)/U^{\text{mod}}(y)$.

Taking the first and second derivatives gives

$$U^{\text{mod}}(y) = U_g(g(y))g_y(y),$$

and

$$U^{\text{mod}}_{yy}(y) = U_{gg}(g(y))(g_y(y))^2 + U_g(g(y))g_{yy}(y).$$

Hence,

$$\gamma \equiv U^{\text{mod}}(y)/U^{\text{mod}}(y) = g_y(y)/g_y(y),$$

Using the definition of $\eta(y) = g_x(y)\gamma(y) - y - (g_x(y)/g_y(y))$, and rearranging gives

$$\eta_i(y) = \gamma \equiv U^{\text{mod}}(y) - \gamma(y).$$

**Proof of Theorem 2.** The following lemma will be useful in our development of Theorem 2.

**Lemma.** $\eta_i(y) = -d(U(y))U_y(y)$, where

$$d(y) = -\frac{m_{yy}(y)}{m_y(U_y(y))}.$$

**Proof.** By definition, $m(t) = U(U^{-1}(t))$. Hence, $U(g(y)) = m(U(y))$. Taking the first and second derivatives with respect to $x$ gives

$$U_x(g(y))g_y(y) = m_{xy}(U(y))U_y(y),$$

$$U_{xy}(g(y))(g_y(y))^2 + U_x(g(y))g_{yy}(y) = m_{xy}(U(y))(U_y(y))^2 + m_x(U(y))U_{yy}(y).$$

Therefore,

$$U_{xy}(g(y))(g_y(y))^2 + U_x(g(y))g_{yy}(y) = m_{xy}(U(y))(U_y(y))^2 + m_x(U(y))U_{yy}(y).$$

By definition of the risk-aversion function, this gives

$$\gamma = \frac{m_{xy}(U(y))}{m_x(U(y))}U_y(y) \equiv \eta_i(y).$$

Q.E.D.
Now we prove the theorem. First, we show invariance under a linear transformation. If \( m(t) \) is the characteristic transformation of \( U, g \), then the characteristic transformation of \( a + bU \) is \( a + bm(t - a)/b \). By direct differentiation, the negative ratio of the second to first derivative of \( a + bm((t - a)/b) \) is \((1/b) \cdot d((t - a)/b) \) and the first derivative of \( a + bU \) is \( bU \). Hence, the product \((1/b)d((t - a)/b)U(x) = d((t - a)/b)U(x) \). Substituting for \( t = a + bU \) gives \( d(U(y))U(y) \), which is independent of both \( a \) and \( b \). Therefore, it is invariant under a linear transformation, and we can assume without loss of generality that \( U > 0 \). Now if \( d(t) \) is negative (\( m(t) \) is a strictly convex function of \( t \)), then \( \eta_y(y) \) is negative, and from Theorem 1, we know that \( g(y) = \hat{g} \). On the other hand, if \( d(t) \) is positive (\( m(t) \) is a strictly concave function of \( t \)), then \( \eta_y(y) \) is positive, and \( g(y) > \hat{g} \). Finally, if \( d(t) = 0 \) is 0 (\( m(t) \) is a linear function of \( t \)), then \( \eta_y(y) = 0 \) and \( g(y) = \hat{g} \).

**Proof of Theorem 3.** We prove this theorem in four steps, assuming small risks where the domain of the lotteries converges to a small neighborhood around the mean. We use the “small \( \sigma \)” notation, \( \alpha(\hat{\delta}) \), which implies that for every \( M > 0 \), there exists \( \delta \) such that \( |(o(\hat{\delta}))| \leq M \delta \delta \rightarrow 0 \), or alternatively that \( \lim_{\delta \rightarrow 0} o(\delta) = 0 \). We also use the “big \( \sigma \)” notation to imply merely the existence of \( N \) such that \( \alpha(\delta) \leq N \delta \delta \rightarrow 0 \).

**Step 1:** First, we prove that

\[
U(g) = m(U(\mu)) + \frac{1}{2} \frac{\partial^2 m(U(\mu))}{\partial \mu^2} \sigma^2 + o(\sigma^2),
\]

as \( (y_{\max} - y_{\min}) \rightarrow 0 \).

**Proof:** We prove this step by taking a Taylor expansion of \( U(g(y)) \) around \( U(g(\mu)) \) to get

\[
U(g(y)) = m(U(y)) = m(U(\mu)) + \frac{\partial m(U(\mu))}{\partial \mu} (y - \mu) + \frac{1}{2} \frac{\partial^2 m(U(\mu))}{\partial \mu^2} (y - \mu)^2 + o(y^2).
\]

This implies that for any positive \( M \), there exists \( \delta > 0 \) such that the residual term satisfies \( o(y^2) \leq M \delta \delta \rightarrow 0 \) on the interval \( y - \mu^2 < \delta \), or, equivalently, on the interval \( y \in [\mu - \sqrt{\delta}, \mu + \sqrt{\delta}] \).

The expected utility of the modified lottery obtained using the Taylor expansion is

\[
U(\tilde{g}) = \int_{y_{\min}}^{y_{\max}} f(y)U(g(y)) dy = \int_{y_{\min}}^{y_{\max}} f(y)m(U(y)) dy + \frac{1}{2} \frac{\partial^2 m(U(\mu))}{\partial \mu^2} \sigma^2 + \int_{y_{\min}}^{y_{\max}} f(y)o(y - \mu^2) dy.
\]

For \( y_{\max} < \mu + \sqrt{\delta}, y_{\min} > \mu - \sqrt{\delta} \), we have \( y \in [\mu - \sqrt{\delta}, \mu + \sqrt{\delta}] \), and the absolute value of the integral on the RHS satisfies

\[
\left| \int_{y_{\min}}^{y_{\max}} f(y)o(y - \mu^2) dy \right| \leq \int_{y_{\min}}^{y_{\max}} f(y)M|y - \mu| dy = M \sigma^2.
\]

By definition, this implies that

\[
\int_{y_{\min}}^{y_{\max}} f(y)o(y - \mu^2) dy = o(\sigma^2), \quad \text{as} \quad (y_{\max} - y_{\min}) \rightarrow 0.
\]

Therefore, (20) can be written as

\[
U(\tilde{g}) = m(U(\mu)) + \frac{1}{2} \frac{\partial^2 m(U(\mu))}{\partial \mu^2} \sigma^2 + o(\sigma^2),
\]

as \( (y_{\max} - y_{\min}) \rightarrow 0 \).

**Now define \( \tilde{x} \) as**

\[
U(\tilde{g}) = U(g(\tilde{x})) \Rightarrow \tilde{x} = g^{-1}(\tilde{g}).
\]

**Step 2:** We prove that

\[
\tilde{x} - \mu = O(\sigma^2), \quad \text{as} \quad (y_{\max} - y_{\min}) \rightarrow 0.
\]

**Proof:** Using the results of Step 1, with the observation that \( m(U(\mu)) = U(g(\mu)) = U^{\text{Mod}}(\mu) \), we get

\[
U^{\text{Mod}}(\tilde{x}) = U^{\text{Mod}}(\mu) + \frac{1}{2} \frac{\partial^2 m(U(\mu))}{\partial \mu^2} \sigma^2 + o(\sigma^2),
\]

as \( (y_{\max} - y_{\min}) \rightarrow 0 \).

This implies that

\[
U^{\text{Mod}}(\tilde{x}) - U^{\text{Mod}}(\mu) = \frac{1}{2} \frac{\partial^2 m(U(\mu))}{\partial \mu^2} \sigma^2 + o(\sigma^2) = O(\sigma^2),
\]

as \( (y_{\max} - y_{\min}) \rightarrow 0 \),

where \( O(\sigma^2) + o(\sigma^2) = O(\sigma^2) \).

Define the function \( F(\cdot) \) as the inverse function of \( U^{\text{Mod}}(\cdot) \). Therefore,

\[
\tilde{x} - \mu = F(U^{\text{Mod}}(\tilde{x})) - F(U^{\text{Mod}}(\mu)).
\]

Expanding \( F(U^{\text{Mod}}(\tilde{x})) \) around the neighborhood of \( F(U^{\text{Mod}}(\mu)) \) gives

\[
\tilde{x} = F(U^{\text{Mod}}(\mu)) + F'(U^{\text{Mod}}(\mu))[U^{\text{Mod}}(\tilde{x}) - U^{\text{Mod}}(\mu)]
\]

\[
+ o(U^{\text{Mod}}(\tilde{x}) - U^{\text{Mod}}(\mu)).
\]

This implies that

\[
\tilde{x} - \mu = F'(U^{\text{Mod}}(\mu))[U^{\text{Mod}}(\tilde{x}) - U^{\text{Mod}}(\mu)]
\]

\[
+ o(U^{\text{Mod}}(\tilde{x}) - U^{\text{Mod}}(\mu)).
\]

Because \( U^{\text{Mod}}(\tilde{x}) - U^{\text{Mod}}(\mu) = O(\sigma^2) \), we have

\[
\tilde{x} - \mu = F'(U^{\text{Mod}}(\mu))O(\sigma^2) + o(o(\sigma^2))
\]

\[
= F'(U^{\text{Mod}}(\mu))O(\sigma^2).
\]

Note that the term \( F'(U^{\text{Mod}}(\mu)) \) is bounded, because

\[
F'(U^{\text{Mod}}(\mu)) = \left( \frac{d}{dx} U^{\text{Mod}}(x) \right)_{x=\mu}^{-1}
\]

\[
= \left( \frac{d}{dg} U(g(x)) \right)_{x=\mu}^{-1}.
\]
and by our previous assumptions \((d/dx)U(x) \neq 0, (d/dx)g(x) > 0\). Therefore,
\[
\tilde{x} - \mu = F(U^*O(\sigma^2)) = O(\sigma^2).
\]

**Step 3:** We prove that
\[
U(\tilde{g}) = m(U(\mu)) + \frac{\partial m(U(\mu))}{\partial y} (\tilde{x} - \mu) + o(\sigma^2),
\]
as \((y_{\text{max}} - y_{\text{min}}) \to 0\).

**Proof.** We first take a Taylor expansion of \(U(g(\tilde{x}))\) around \(U(g(\mu))\) to get
\[
U(\tilde{g}) = U(g(\mu)) + \frac{\partial U(g(\mu))}{\partial y} (\tilde{x} - \mu) + o(\tilde{x} - \mu)).
\]

Then we use the results of Step 2 and observe that
\[
\tilde{x} - \mu = O(\sigma^2)
\]
to get
\[
U(\tilde{g}) = m(U(\mu)) + \frac{\partial m(U(\mu))}{\partial y} (\tilde{x} - \mu) + o(\sigma^2).
\]

**Step 4:** We now use Steps 1, 2, and 3 to prove the theorem. Equating the expressions of \(U(\tilde{g})\) from Steps 1 and 3, and rearranging, gives
\[
\tilde{x} = \mu + \frac{\sigma^2}{2} (1/2)(\partial^2/\partial y^2)m(U(\mu)) + o(\sigma^2),
\]
as \((y_{\text{max}} - y_{\text{min}}) \to 0\). (22)

We have
\[
\frac{\partial m(U(\mu))}{\partial y} = \frac{\partial m(U(\mu))}{\partial y} \bigg|_{y=\mu}
\]
and
\[
\frac{\partial^2 m(U(\mu))}{\partial y^2} = 
\frac{\partial^2 m(U(\mu))}{\partial y^2} \bigg|_{y=\mu} + \frac{\partial^2 U(y)}{\partial y^2} \bigg|_{y=\mu}.
\]
Dividing gives
\[
\frac{\partial^2 m(U(\mu))}{\partial y^2} = 
\frac{\partial^2 m(U(\mu))}{\partial y^2} \bigg|_{y=\mu} + \frac{\partial^2 U(y)}{\partial y^2} \bigg|_{y=\mu}.
\]

Hence,
\[
\tilde{x} = \mu - \frac{\sigma^2}{2} \left[ \eta_\gamma(\mu) + \gamma(\mu) \right] + o(\sigma^2),
\]
as \((y_{\text{max}} - y_{\text{min}}) \to 0\).

By definition, \(\tilde{x} = g^{-1}(\tilde{g})\). Therefore,
\[
\tilde{g} = g \left( \mu - \frac{\sigma^2}{2} \left[ \eta_\gamma(\mu) + \gamma(\mu) \right] + o(\sigma^2) \right),
\]
as \((y_{\text{max}} - y_{\text{min}}) \to 0\).

However, from (22) for the special case of \(g(t) = t\), we get
\[
\tilde{g} = \mu - \frac{\sigma^2}{2} \gamma(\mu) + o(\sigma^2),
\]
as \((y_{\text{max}} - y_{\text{min}}) \to 0\).

Therefore,
\[
\tilde{g} = g \left( \tilde{y} - \frac{\sigma^2}{2} \eta_\gamma(\mu) + o(\sigma^2) \right),
\]
as \((y_{\text{max}} - y_{\text{min}}) \to 0\).

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