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SUBSIDIZED SECURITY AND STABILITY OF EQUILIBRIUM SOLUTIONS IN AN N-PLAYER GAME WITH ERRORS

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Subsidized Security and Stability of Equilibrium Solutions in an N-Player Game with Errors

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Abstract

Optimizing resource allocation in interdependent security problems is a serious challenge for U.S. homeland security. In this paper, game theory is applied to this challenge in the case where investment by one defender has positive externalities for other players. The phenomena of tipping and cascading are discussed, and we explore how to target subsidized security in order to achieve the best results from tipping. In particular, we examine the effects of subsidized security on both the stability of the equilibrium solutions and the total social costs. Results indicate, not surprisingly, that equilibrium solutions are less stable in systems with larger probabilities of agent error, but also (perhaps less obviously) in systems with larger numbers of agents. We show that subsidization of security investments can increase the stability of equilibrium solutions, and also decrease the total expected social costs. The above findings are illustrated in a numerical example.

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1. Introduction

After September 11, 2001, homeland security has received a great deal of attention in the U.S. Since security-related resources are always limited, maximizing security subject to limited resources is a key challenge. “Tipping” (Schelling, 1978; Gladwell, 2000) has been suggested as a cost-effective way to encourage security investment, since if successful, a subsidy or other incentive to encourage a relatively small number of agents to invest can induce other agents to also invest in security, and therefore more nearly achieve the social optimum.

Many security problems (including aviation security, computer security, and supply-chain security) involve interdependence among potential defenders, meaning that one agent’s strategy can affect the security environment for other agents. For example, poor security on the part of one airline, computer user, or supply-chain partner can increase the rate of attacks on other agents.

Game theory has already been applied to such interdependent security problems (Kunreuther and Heal, 2003; Bier and Gupta, 2005). However, to our knowledge, no previous studies have investigated the effect of subsidizing of security investment on the stability and total social cost of the equilibrium solutions. Secondly, previous studies have not investigated the effect of error-prone agents on the stability of equilibrium solutions, and how subsidization of security can be used to minimize the adverse effects of such errors. Finally, previous studies have not investigated how the total social costs in the case of subsidized security investment compare to the total costs in the absence of subsidization.

The next section of this paper formulates a general interdependent security model for an arbitrary number of agents with attacks occurring over time. Section 3 solves this model and gives the dominant strategies and equilibrium solutions, focusing on the case of agents with heterogeneous time preferences. Section 4 discusses the phenomena of tipping and cascading in this context, and determines the minimal number of agents that must receive subsidized security in order to cause tipping. We also discuss which agents should be targeted in order to maximize the beneficial effects of tipping and cascading.

For simplicity, we then focus on the case of agents with homogeneous discount rates. In this context, Section 5 studies the stability of equilibrium solutions and explores whether subsidizing security investment can increase the stability of the socially optimum solution. Section 6 studies the effects of erroneous choices on the equilibrium solutions and whether the subsidizing security investment can reduce the adverse effects of erroneous choices. Section 7 studies the relationship between subsidization of security investment and the total expected social cost, to determine when subsidies can help minimize total costs. Finally, Section 8 summarizes the previous sections and discusses the policy implications of our work.

2. Model Formulation

Our basic model allows both direct attacks on an agent, and also indirect attacks (e.g., contamination from another agent in the system), as illustrated in Fig. 1. As in Bier and Gupta (2005), we assume that the time, t, of a direct attack on any of N agents follows an exponential distribution, and may result in an indirect attack on one or more other agents with some probability. Like both Kunreuther and Heal (2003) and Bier and Gupta (2005), we assume that an attack on any agent is catastrophic, so that subsequent attacks on the same agent are not considered. For $i = 1\ldots N$, we define the system parameters as follows:

- λ_i = rate of direct attacks on agent *i*.
- $\tilde{\lambda}_i$ = total rate of all attacks on agent *i* (including indirect attacks).
- q_{ij} = probability that an attack on agent *i* infects agent *j* where we define $q_{ii} = 1$.
- r_i = discount rate of agent *i*.
- L_i = loss suffered by agent *i* if it is attacked, either directly or indirectly.
- C_i = cost of investing in security for agent *i*. This investment is assumed to eliminate the risk of direct attacks, but have no effect on the risk of infection by indirect attacks from other agents. We assume that $0 < C_i < L_i \forall i = 1,...,N$.
- s_i = investment strategy for agent *i*, where $s_i = 1$ if agent *i* invests in security and $s_i = 0$ if agent *i* does not invest.
- $s_{-i} = \{s_j, j \neq i\}$, the strategies of other agents.
- $P_i(s_i, s_{-i})$ = total expected cost borne by agent *i* when it chooses strategy s_i (including both the cost of investment, if any, and the expected loss due to attacks), given the strategies of the other agents.

The expected loss experienced by agent *i* due to attacks is given by $L_i \int_0^\infty f_i(t)\exp(-r_i t)dt$, where $f_i(t) = \tilde{\lambda}_i \exp(-\tilde{\lambda}_i t)$ is the probability density function for the time of the first attack on agent *i*, and $\tilde{\lambda}_i = (1-s_i)\lambda_i + \sum_{j=1, j\neq i}^{N}(1-s_j)q_{ji}\lambda_j$ is the total rate of attacks against agent *i*. Hence, the net present value of the expected loss due to attacks experienced by agent *i* is

$$E(Loss) = L_i \int_0^\infty \tilde{\lambda}_i \exp(-\tilde{\lambda}_i t - r_i t)dt = L_i / (1 + r_i / \tilde{\lambda}_i) \qquad (1)$$

The total expected cost to agent i is given by

$$P_i(s_i, s_{-i}) = s_i C_i + L_i / (1 + r_i / \tilde{\lambda}_i) \quad (2)$$

3. Equilibrium Solutions

Definition 1: Nash equilibrium in our model is a set of strategies $\{s_i, i = 1, ..., N\}$ such that no one agent would be better off by switching strategies unless at least one other agent also switched. Thus, at equilibrium, we must satisfy the following system of inequalities:

$$P_i(s_i, s_{-i}) < P_i(1 - s_i, s_{-i}) \forall i = 1, ..., N \quad (3)$$

For simplicity, in the remainder of this paper, we consider the case where only the discount rates (r_i) of the agents differ. In other words, we let $\lambda_i = \lambda$, $C_i = C$, $L_i = L$, and $q_{ij} = q$ for i, j=1, 2,…, N, $i \neq j$, since the effects of these parameters have already been investigated by Kunreuther and Heal (2003). Thus, equation (2) becomes:

$$P_i(s_i, s_{-i}) = s_i C_i + L / (1 + r_i / \tilde{\lambda}) \quad (4)$$

where $\tilde{\lambda} = (1 - s_i)\lambda + \sum_{j=1, j \neq i}^{N} (1 - s_j) q \lambda$.

Suppose that in some equilibrium solution, exactly M agents choose to invest ($0 \leq M \leq N$). Then there are three possible cases:

(1) All agents choose to invest (M=N);

(2) Some agents choose to invest and some not (0<M<N); and

(3) No agents choose to invest (M=0).

Table 1 gives the costs to both investing and non-investing agents in all three cases. It also specifies what the costs would be to any given agent that deviates from the equilibrium action s_i, conditional on the other agents' strategies; i.e. $P_i(1-s_i, s_{-i})$. For convenience, in Table 1 and throughout the remainder of this paper, we have renumbered the agents so that the first M agents choose to invest, and the remaining N-M agents choose not to invest (i.e., $s_i = 1 \;\; \forall i = 1,...,M$, and $s_i = 0 \;\; \forall i = M+1,...,N$). Solving the system of inequalities (3) using the costs in Table 1 yields conditions for the discount rates at which agents will be willing to invest in security at equilibrium. These conditions are summarized in Table 2, using the following notation:

- $\tilde{N} = \max\left\{ n \mid n \le C(L/C-1)^2/(4Lq), n \in Z^+ \right\}$;
- $R_1(k) = \lambda\left[L/C - 1 - 2kq - \sqrt{\left(L/C-1\right)^2 - 4qkL/C} \right]/2$, k=0… $\tilde{N}$; and
- $R_2(k) = \lambda\left[L/C - 1 - 2kq + \sqrt{\left(L/C-1\right)^2 - 4qkL/C} \right]/2$, k=0… $\tilde{N}$.

Here, $\tilde{N}$ is a bound on the number of agents there can be in a system for certain properties to hold, and $R_1(k)$ and $R_2(k)$ are bounds on an agent's discount rate. By taking derivatives, it is easy to show that $R_1(k)$ and $R_2(k)$ are increasing and decreasing in k, respectively. Also, note that $R_1(\tilde{N}) < R_2(\tilde{N})$, and $R_1(0)=0$. Thus, the following relationship holds, as shown in Fig. 2:

$$0 = R_1(0) < R_1(1) < \ldots < R_1(\tilde{N}) < R_2(\tilde{N}) < \ldots < R_2(1) < R_2(0) \qquad (5)$$

Definition 2: The partition $\{S_l,\ l=0\ldots\min\{N, \tilde{N}+1\}\}$ of the domain of discount rates is defined as follows:

$$S_0 = (-\infty, R_1(0)] \cup [R_2(0), +\infty),$$

$$S_l = (R_1(l-1), R_1(l)] \cup [R_2(l), R_2(l-1)) \;\; \forall l = 1,..., \min\{N\text{-}1, \tilde{N}\}, \text{ and}$$

$$S_l = (R_1(l-1), R_2(l-1)),\ l = \min\{N, \tilde{N}+1\}.$$

This partition is represented graphically in Fig. 2 for the case where $N \le \tilde{N}$. When $N > \tilde{N}$, $R_1(k)$ and $R_2(k)$ are not defined for $k > \tilde{N}$, so S_l is empty for $l > \tilde{N} + 1$.

Definition 3: Given a system in which exactly *M* agents invest at equilibrium:

(1) Let *Inv*(*M*) be the set of possible discount rates for the *M* investing agents;

(2) Let *Non*(*M*) be the set of possible discount rates for the *N*-*M* non-investing agents; and

(3) Let $Cmpl(M) = [Inv(M) \cup Non(M)]^C$ be the set of discount rates that can not be held by any agent if exactly *M* agents invest at equilibrium.

Using the notation above, Table 2 can be rewritten as shown in Table 3, specifying the ranges of discount rates possible for both the investing and non-investing agents, given the number of agents *M* investing at equilibrium. These ranges are also illustrated in Figures 3 and 4, for the investing and non-investing agents, respectively. (As before, when $N > \tilde{N}$, some of the S_l will be empty.) Note also that the existence of an equilibrium strategy with *M* investing agents implies that no agent has a discount rate in the set $Cmpl(M)$.

Definition 4: Strategy s_i is a *dominant strategy* for agent *i* if and only if $P_i(s_i, s_{-i}) < P_i(1 - s_i, s_{-i})$ for all s_{-i}.

Theorem 1: If $r_i \in S_N$ for some agent *i*, then $s_i = 1$ is a dominant strategy for agent *i* (i.e., that agent will prefer to invest in security, regardless of the decisions of other agents). Conversely, if $r_i \in S_0$ for some agent *i*, then $s_i = 0$ is a dominant strategy for that agent

Proof: It is possible to show that at least one equilibrium solution always exists (proof omitted for reasons of space). Therefore, we can use Table 3 to characterize the possible equilibrium solutions. If $r_i \in S_N$ for some agent *i*, then according to Table 3, not investing cannot be an equilibrium strategy for this agent. Therefore, investment must be a dominant strategy. A similar argument holds for the second part of the theorem.

Theorem 2: Holding all else constant, when the number of agents in the system (*N*) increases, the range of discount rates S_N for which investing is a dominant strategy becomes smaller. If $N > \tilde{N} + 1$, then there is no discount rate for which investing in security is a dominant strategy. The range of discount rates S_0 for which not investing is a dominant strategy does not depend on *N*.

Proof: Since $R_1(k)$ and $R_2(k)$ are increasing and decreasing in *k* for $k=1\ldots\tilde{N}$, respectively, the set $S_N = (R_1(N-1), R_2(N-1))$ becomes smaller as *N* increases. For $N > \tilde{N} + 1$, the set S_N is empty. The set $S_0 = (-\infty, R_1(0)] \cup [R_2(0), +\infty)$ does not depend on *N*, by the definitions of $R_1(k)$ and $R_2(k)$.

Remark: This theorem implies that for systems with sufficiently large numbers of agents, investing in security will not be a dominant strategy for any agent. Agents can of course still choose to invest, but will do so in equilibrium only if other agents also invest.

Example 1: We use the following parameters to illustrate the results discussed in Theorem 2: *C*=10; *L*=1000; *q*=0.01; λ=0.01; *N*=2000; and therefore $\tilde{N} = 2450$. Fig. 5 shows the regions of discount rates in which investing and not investing, respectively, are dominant strategies, as a function of the number of agents.

The remainder of this paper focuses on the case where no agent has a dominant strategy. In other words, we assume that the discount rate of agent *i* satisfies $r_i \notin S_0$ and $r_i \notin S_N \forall i = 1, ..., N$.

Theorem 3: In an *N*-agent model, if no agent has a dominant strategy, there exist at least two equilibrium solutions: {all invest}; and {none invest}. The total cost borne by any given agent in the equilibrium {all invest} is lower than the corresponding cost in the equilibrium {none invest}, with the difference between these two costs growing with *N*.

Proof: From Table 3, we have $r_i \in Inv(N)$ and $r_i \in Non(0)$, which implies that {all invest} and {none invest} are both equilibrium solutions. The total cost for agent *i* equals C for the equilibrium {all invest}, and $L / \left[1 + r_i / \left(\lambda + (N-1)q\lambda\right)\right]$ for the equilibrium {none invest}. By the definitions of $R_2(0)$ and S_l, we can show that $r_i < R_2(0)$ and

$$C = L / \left[1 + R_2(0) / \lambda\right] < L / \left[1 + r_i / \left(\lambda + (N-1)q\lambda\right)\right] \ \forall i = 1,..., N \tag{6}$$

So, the total cost borne by each agent in the equilibrium {all invest} is lower than the corresponding cost in the equilibrium {none invest}. Furthermore, as *N* grows (assuming that all other parameters are held constant), straightforward algebra shows that the cost difference between the left and right sides of inequality (6) will increase.

Theorem 4: In a homogeneous *N*-agent model in which the agents have no dominant strategy, there exist only two equilibrium solutions: {all invest}; and {none invest}.

Proof: Theorem 3 indicates that {all invest} and {none invest} are both equilibrium solutions. Suppose that there exists an equilibrium with M investing agents such that $0<M<N$. Then, according to the definition of equilibrium, we must have $r_i \in Inv(M)$ $\forall i =1\ldots M$ and $r_i \in Non(M)$ $\forall i =M+1\ldots N$. However, homogeneity implies that $r_i = r$ $\forall i =1\ldots N$, and we know that $Inv(M) \bigcap Non(M) = \phi$. Therefore, a contradiction has been found.

4. Tipping and Cascading

We now discuss the possibility of tipping, and its effect on the equilibrium solution. In general, starting with an equilibrium in which M agents invest (for $M<N$), ensuring that some additional agents invest will tend to make investing more attractive for the remaining agents. In practice, there may be different ways of ensuring that agents invest, such as mandating investment in security, or providing subsidized (free) security. In this paper, we focus on the latter case.

Definition 5: Given a system in which exactly M agents invest at equilibrium, suppose that h additional agents receive subsidized (free) security from a third party ($0 \le h \le N - M$). Let $Inv_c(M, h)$ be the set of possible discount rates for those agents who did not invest in the initial equilibrium, but would find investing attractive if the number of investing agents increased from M to $M+h$. Note that this will include not only agents with discount rates in $Inv(M+h)$, but also those with discount rates in $Cmpl(M+h)$, since there can be an equilibrium strategy exactly with $M+h$ investing agents only if no agents have discount rates in the set $Cmpl(M+h)$. Thus, we will have:

$$Inv_c(M, h) = Inv(M+h) - Inv(M) \bigcup Cmpl(M+h) \tag{7}$$

Here, $-$ is the set operator representing the difference between sets $Inv(M+h)$ and $Inv(M)$. Let $\Theta(S)$ be the cardinality of the set *S*. Then $\Theta\left[Inv_c(M,h)\right]$ is the number of agents that could be induced to invest by tipping.

Remark: It is straightforward to show that $\Theta\left[Inv_c(M,h)\right]$ is non-decreasing in *h* for any given value of *M*, with $\Theta\left[Inv_c(M,0)\right]=0$.

Starting with an equilibrium in which exactly *M* agents invest, the number of agents *h* that must receive subsidized security in order to lead to tipping must satisfy $\Theta\left[Inv_c(M,h)\right]>0$. If $\Theta\left\{Inv_c\left[M+h,\Theta\left[Inv_c(M,h)\right]\right]\right\}>0$, then additional agents will to invest when the number of investing agents increases from $M+h$ to $M+h+\Theta\left[Inv_c(M,h)\right]$, and so on. We call this phenomenon "cascading."

Note that no tipping or cascading will occur if $\Theta\left[Inv_c(M,h)\right]=0$. The minimal number of agents that must receive free security in order to lead to tipping is given by $\min\left\{h \mid \Theta\left[\left(Inv_c(M,h)\right)\right]>0\right\}$. The discount rates of the subsidized agents are irrelevant to determining whether tipping occurs, since this depends only on the total number of agents receiving subsidized (free) security. However, the discount rates of the subsidized agents do determine whether cascading occurs, and how far it progresses. Therefore, it makes sense to target any subsidies at those agents who are least likely to begin investing due to tipping (e.g., agents with discount rates in or near the region where not investing is the dominant strategy). These arguments are illustrated in the example below; see also Kunreuther and Heal (2003) and Dixit (2003).

Example 2: Consider an N-agent system in which $N< \tilde{N} +1$ and $r_i \in S_{i-1} \forall i =1\ldots N$. Perusal of Fig. 3 indicates that this example, there exists no equilibrium with a non-zero number of investing agents. Furthermore, Fig. 4 shows that {none invest} with M=0 investing agents is an equilibrium for this example. Now suppose that agent 1 receives free security. Then agent N will choose to invest, because $r_N \in S_{N-1} \subset Inv_c(0,1)$. In other words, since agent 1 will invest for sure, not investing is no longer optimal for agent N (from Fig. 4). Since agent N will be better off investing, there will now be M=2 investing agents. Therefore, agent N-1 will also begin investing because $r_{N-1} \in S_{N-2} \subset Inv_c(1, \Theta\left[Inv_c(0,1)\right])$. Similarly, agent N-2 will begin investing once there are M=3 investing agents, and so on. Thus, if agent 1 receives free security, {all invest} will become the unique equilibrium. Note also that the discount rate of the subsidized agent determines how many agents will decide to invest as a result of cascading. In this example, it is straightforward to see that if agent i is the one that receives free security, the system will end up in a unique equilibrium with $M=i$ investing agents.

5. Stability of Equilibrium Solutions

For simplicity, we now consider the case of homogeneous time preferences in an N-agent model in which no agent has a dominant strategy. In other words, we let $r_i = r \in S_k \forall i =1\ldots N$ for $0<k<\min\{N, \tilde{N} +1\}$.

Definition 6: In an N-agent homogeneous model, let n be the greatest integer such that, even if n agents all change to the opposite strategy, the remaining N-n agents will still not want to change their strategies. We then define the stability level of an equilibrium (either {all invest} or {none invest}) to be $\alpha = n/(N-1)$.

Remark: If $\alpha = 0$, then the corresponding equilibrium is completely unstable; that is, if even one agent changes strategy, then at least one other agent will also prefer to change strategy. If $\alpha = 1$, the corresponding equilibrium is completely stable; no matter how many agents change strategies, no other (rational) agent would want to change strategy. Note that the stability of Nash equilibrium solutions has been defined variously in other research; see for example Damme (1991), Kohlberg and Mertens (1986), and Okada (1981).

Let h be the number of agents receiving subsidized (free) security, such that $0 \le h \le N$. Since the h agents receiving subsidized (free) security need not incur any cost to invest, we consider only the strategies of the N-h non-subsidized agents. Let {all invest}$_{N\text{-}h}$ and {none invest}$_{N\text{-}h}$ be sub-equilibrium solutions describing the possible behavior of these N-h agents.

Theorem 5: Consider a model with N homogeneous agents, where $r_i = r \in S_k \forall i = 1 \ldots N$ for $0 < k < \min\{N, \tilde{N} + 1\}$, and assume that a third party offers subsidized (free) security to h agents. Then, {all invest}$_{N\text{-}h}$ will be a sub-equilibrium for the N-h non-subsidized agents for any value of h. This sub-equilibrium has stability $\alpha = (k-1)/(N-h-1)$ if $h \le N-k-1$, and $\alpha = 1$ if $h \ge N-k$. By contrast, {none invest}$_{N\text{-}h}$ is a sub-equilibrium only if $h \le N-k-1$, in which case its stability is given by $\alpha = (N-h-k-1)/(N-h-1)$.

Proof: If $h \le N-k-1$, then after k-1 agents change from investing (in the sub-equilibrium {all invest}$_{N\text{-}h}$) to not investing, {all invest}$_{N\text{-}h\text{-}k+1}$ is still a sub-equilibrium for the remaining N-h-k+1 agents, because $r \in S_k \subset Inv(N-k+1)$. However, after k agents change from investing to not investing, {all invest}$_{N\text{-}h\text{-}k}$ is no longer a sub-equilibrium for the remaining N-h-k agents, because $r \in S_k \not\subset Inv(N-k)$. So, n=k-1 is the largest number of agents that can change strategies such that the remaining agents will want to continue investing at sub-equilibrium. Therefore, {all invest}$_{N\text{-}h}$ has stability $\alpha = (k-1)/(N-h-1)$. Similarly, it can be shown that {none invest}$_{N\text{-}h}$ has stability $\alpha = (N-h-k-1)/(N-h-1)$ if $h \le N-k-1$. Now we consider the case $h \ge N-k$. In this case, {all invest}$_{N\text{-}h}$ is a sub-equilibrium for the non-subsidized *N-h* agents, and has stability $\alpha = 1$ (since for $h \ge N-k$, it is no longer possible to have *k* of the *N-h* non-subsidized agents change strategies). To see why {none invest}$_{N\text{-}h}$ is not a sub-equilibrium in this case, note that if all of the *N-h* non-subsidized agents choose not to invest, there will be only *M*=*h* investing agents. From Table 3, we know that {none invest}$_{N\text{-}h}$ will be a sub-equilibrium for the *N-h* non-subsidized agents if and only if $r \in S_k \subset Non(h)$, or equivalently $h \le N-k-1$.

Remark: If $r_i = r \in S_k \forall i$ =1…*N*, the stability of the sub-equilibrium {all invest}$_{N\text{-}h}$ is increasing in *h* for $h \le N-k-1$, , and equals 1 (i.e., completely stable) when $h \ge N-k$ agents receive subsidized (free) security.

Theorem 6: If both {all invest}$_{N\text{-}h}$ and {none invest}$_{N\text{-}h}$ are possible sub-equilibrium solutions, then {all invest}$_{N\text{-}h}$ will be more stable than {none invest}$_{N\text{-}h}$ when $k > (N-h)/2$, or equivalently when $r \in \bigcup_{l=\lfloor (N-h)/2 \rfloor +1}^{\min\{\tilde{N}+1, N-1\}} S_l$. (Here, $\lfloor x \rfloor$ is the floor function, giving the greatest integer less than or equal to *x*.) Conversely, {none invest}$_{N\text{-}h}$ will be more stable than {all invest}$_{N\text{-}h}$ when $k < (N-h)/2$, or equivalently when $r \in \bigcup_{l=1}^{\lfloor (N-h)/2 \rfloor} S_l$. If *N-h* is even, then the two sub-equilibrium solutions will be equally stable when $k = (N-h)/2$, or $r \in S_{(N-h)/2}$.

Proof: Follows directly Theorem 5.

Remark: Generally, {all invest}$_{N\text{-}h}$ will tend to be more stable than {none invest}$_{N\text{-}h}$ when the discount rate of the (homogeneous) agents is close to the region where investing is dominant (S_N). Similarly, {none invest}$_{N\text{-}h}$ will be more stable than {all invest}$_{N\text{-}h}$ when the discount rate is close to the region where not investing is dominant (S_0). If N-h is even, then there exists a middle range ($S_{(N-h)/2}$) where the two sub-equilibrium solutions are equally stable. This is illustrated in Fig. 6 for the case h=0.

6. Erroneous Choice

In a model with N homogeneous agents, where $r \in S_k$ for $0<k<N$, we have shown that the equilibrium {all invest} is the social optimum, and moreover has lower cost than the equilibrium {none invest} for any given agent individually. Therefore, it may be reasonable to expect that any rational agent would choose to invest in this case. However, in practice, some agents may choose not to invest even when it would be in their interest to do so. We denote such behavior an *erroneous choice*. In this section, rather than assuming that all of the N (homogenous) agents make the same choice (as in Section 5), we assume that some agents (at random) erroneously choose not to invest, and the remaining agents choose whichever strategy will be the social optimum in light of the number of erroneous choices. We here examine the effect of erroneous choice on the sub-equilibrium solutions for the remaining agents who do not make errors. We also examine how subsidization of security investment can be used to help counteract any adverse effect of erroneous choices.

Let h be the number of agents receiving subsidized (free) security, and let x be the number of agents making erroneous choices, such that $0 \le x \le N-h$. Since the h agents receiving subsidized (free) security need not incur any cost to invest, we consider only the strategies of the N-h-x non-subsidized agents who do not make erroneous choices. Let {all invest}$_{N\text{-}h\text{-}x}$ and {none invest}$_{N\text{-}h\text{-}x}$ be sub-equilibrium solutions describing the possible behavior of these *N*-*h*-*x* agents.

Theorem 7: If subsidized (free) security is provided to $h \geq N-k$ agents, then no matter how many agents *x* make erroneous choices, {all invest}$_{N\text{-}h\text{-}x}$ will be the unique sub-equilibrium for those non-subsidized agents not making erroneous choices. Similarly, if $x \geq k$ agents make erroneous choices, then {none invest}$_{N\text{-}h\text{-}x}$ will be the unique sub-equilibrium for those non-subsidized agents not making erroneous choices. If $x+1 \leq k \leq N-h-1$, then {all invest}$_{N\text{-}h\text{-}x}$ and {none invest}$_{N\text{-}h\text{-}x}$ are both possible sub-equilibrium solutions. In this case, the total cost borne by any of the *N* agents individually in {all invest}$_{N\text{-}h\text{-}x}$ (when *h* agents receive subsidized security, *x* agents make erroneous choices, and *N*-*h*-*x* agents invest) is lower than the corresponding cost when the *N*-*h*-*x* agents do not invest. This implies that {all invest}$_{N\text{-}h\text{-}x}$ is the socially optimal sub-equilibrium.

Proof: Since the agents are assumed to be homogeneous, by Theorem 4 we know that all of the non-subsidized *N*-*h*-*x* agents not making erroneous choices will choose the same strategy in any sub-equilibrium. If all of these *N*-*h*-*x* agents choose to invest, then there will be a total of *M*=*N*-*x* investing agents (including the *h* agents receiving free security). From Table 3, we know that {all invest}$_{N\text{-}h\text{-}x}$ will be a sub-equilibrium for the *N*-*h*-*x* non-subsidized agents not making erroneous choices if and only if $r \in S_k \subset Inv(N-x)$, or equivalently

$$k \geq x+1 \tag{8}$$

In this case, Table 1 indicates that the total cost borne by each of the agents receiving subsidized (free) security is given by $C_{Sub1} \equiv L/\left[1+r/(xq\lambda)\right]$; the total cost borne by each of the non-subsidized *N*-*h*-*x* agents not making erroneous choices is given by $C_{Inv} = C + L/\left[1+r/(xq\lambda)\right]$, where *C*>0 is the cost for any one agent to invest in security; and the total cost borne by each of the x agents making erroneous choices is given by $C_{Err} \equiv L/\left[1+r/(\lambda+(x-1)q\lambda)\right]$.

Similarly, if all of the remaining N-h-x agents choose not to invest, then there will be only M=h investing agents. Again, from Table 3, {none invest}$_{N\text{-}h\text{-}x}$ will be a sub-equilibrium for these N-h-x agents if and only if $r \in S_k \subset Non(h)$, or equivalently

$$k \leq N-h-1 \tag{9}$$

In this case, Table 1 shows that the total cost borne by each of the agents receiving free security is given by $C_{Sub2} \equiv L/\left[1+r/\left((N-h)q\lambda\right)\right]$, and the total cost borne by any of the N-h non-subsidized agents is given by $C_{Non} \equiv L/\left[1+r/\left(\lambda+(N-h-1)q\lambda\right)\right]$.

There will always exist at least one sub-equilibrium, because at least one of inequalities (8) and (9) will hold (by the assumption that $0 \leq x \leq N-h$, and the fact that x, h, k, and N are all integers). If $h \geq N-k$, then {none invest}$_{N\text{-}h\text{-}x}$ will not be a possible sub-equilibrium solution, so {all invest}$_{N\text{-}h\text{-}x}$ will be the unique sub-equilibrium for all values of $x \leq N-h$. Conversely, if $x \geq k$, then {all invest}$_{N\text{-}h\text{-}x}$ will not be a sub-equilibrium, so {none invest}$_{N\text{-}h\text{-}x}$ will be the unique sub-equilibrium for all values of $h \leq N-k-1$. Finally, if $x+1 \leq k \leq N-h-1$, then {all invest}$_{N\text{-}h\text{-}x}$ and {none invest}$_{N\text{-}h\text{-}x}$ will both be sub-equilibrium solutions. In this case, we will have $C_{Inv} < C_{Non}$, $C_{Err} \leq C_{Non}$ and $C_{Sub1} \leq C_{Sub2}$ (proof omitted for reasons of space). Thus, the costs borne by any of the N agents individually in the sub-equilibrium {all invest}$_{N\text{-}h\text{-}x}$ will be less than or equal to the corresponding costs in the sub-equilibrium {none invest}$_{N\text{-}h\text{-}x}$.

Remark: If those non-subsidized agents not making erroneous choices always choose the social optimum, then they will choose to invest whenever the sub-equilibrium {all invest}$_{N\text{-}h\text{-}x}$ exists.

Theorem 8: Suppose that each non-subsidized agent independently makes an erroneous choice with probability ε, where $0 \le \varepsilon \le 1$. In this case, the number of agents X making erroneous choices is a random variable with binomial probability mass function given by $P(X = x) = \binom{N-h}{x} \varepsilon^x (1-\varepsilon)^{N-h-x}$. Let P_{Inv} be the probability that {all invest}$_{N\text{-}h\text{-}X}$ is a sub-equilibrium for those non-subsidized agents not making erroneous choices. Then, we have $P_{Inv} = 1$ if $h \ge N-k$, and $P_{Inv} = \sum_{x=0}^{k-1} \binom{N-h}{x} \varepsilon^x (1-\varepsilon)^{N-h-x}$ if $h \le N-k-1$.

Proof: From Theorem 7, if $h \ge N-k$, then we must have $P_{Inv} = 1$, since {all invest}$_{N\text{-}h\text{-}x}$ is the unique sub-equilibrium for any value of x in that case. From inequality (8), if $h \ge N-k$, then the probability that {all invest}$_{N\text{-}h\text{-}X}$ is a sub-equilibrium is given by $P_{Inv} = P(X \le k-1)$.

Remarks: If fewer than *N*-*k* agents receive subsidized (free) security, then P_{Inv} will be increasing in the number of agents *h* receiving free security (all else constant), in part because provision of free security to a larger number of agents reduces the maximum possible number of agents who could make erroneous choices. P_{Inv} is also increasing in *k*, where $r \in S_k$ is the discount rate of the (homogeneous) agents; that is, as *r* gets closer to the region where investing is dominant (all else constant), it becomes more likely that investing will be a sub-equilibrium for the *N*-*h*-*X* non-subsidized agents not making erroneous choices. All else constant, P_{Inv} is also decreasing in both the error probability ε and the number of agents *N*.

The above observations are based on known properties of the binomial distribution (Bickel and Doksum, 2001).

7. Total Social Cost

Let $C_F(h)$ be the cost to a third party of providing subsidized (free) security to h agents, $C_A(h)$ be the total (expected) costs actually paid by the N agents, and $C_S(h) = C_F(h) + C_A(h)$ be the total (expected) social cost. In the case described in Section 5 (where all non-subsidized agents make the same choice), Theorem 4 shows that there are only two possible sub-equilibrium solutions. For the sub-equilibrium {all invest}$_{N-h}$, we have

$$C_A(h) = (N-h)C \tag{10}$$

For {none invest}$_{N-h}$, we have

$$C_A(h) = hC_{Sub2} + (N-h)C_{Non} \tag{11}$$

Here, C_{Sub2} and C_{Non} are as defined in the proof of Theorem 7. In the case described in Section 6 (involving erroneous choices), if $h \geq N-k$, then we have

$$C_A(h) = \sum_{x=0}^{N-h} P(X=x)\left[(N-h-x)C_{Inv} + xC_{Err} + hC_{Sub1}\right] \tag{12}$$

Similarly, if $h \leq N-k-1$,

$$C_A(h) = \sum_{x=0}^{k-1} P(X=x)\left[(N-h-x)C_{Inv} + xC_{Err} + hC_{Sub1}\right] + (1-P_{Inv})\left[(N-h)C_{Non} + hC_{Sub2}\right] \tag{13}$$

Here, C_{Inv}, C_{Err}, C_{Non}, C_{Sub1}, and C_{Sub2} are as defined in the proof of Theorem 7, and P_{Inv} and $P(X=x)$ are as given in Theorem 8.

Theorem 9 As defined in equalities (10)-(13), the total cost $C_A(h)$ borne by agents is non-increasing in the number of agents receiving subsidized (free) security; i.e., agents will in general benefit from subsidized (free) security.

Proof: Omitted for reasons of space.

We now explore the effect of providing subsidized (free) security to a subset of agents on the total (expected) social cost, by considering three possible functional forms for $C_F(h)$, all of which satisfy $dC_F(h)/dh \leq C$ (where for simplicity, we treat *h* as a continuous variable):

(1) $C_F(h) = Ch$; i.e., the cost to a third party of providing security is the same as the cost to the agents themselves.

(2) $C_F(h)$ increasing and concave, with $dC_F(h)/dh \leq C \forall h$; i.e., a third party can provide security at lower cost than the individual agents could (e.g., due to economies of scale).

(3) $C_F(h)$ constant in *h*. This is a bounding case, in which once security technology (e.g., anti-virus software) has been developed, it can be provided to any number of agents at no additional cost. In this case, it would be clearly be optimal to give free security to all agents, provided that the constant $C_F(h)$ is sufficiently small relative to $C_A(0)$.

Example 3: In this example, we now numerically explore the effects of offering subsidized (free) security for the case discussed in Section 5 (where all non-subsidized agents make the same choice), using the following parameters: *C*=10; *L*=1000; *q*=0.01; λ=0.01; *k*=1200; and *N*=2000. Fig. 7 shows the stability of {all invest}$_{N-h}$ and {none invest}$_{N-h}$ as a function of *h*. Providing subsidized (free) security increases the stability of {all invest}$_{N-h}$ and decreases the stability of {none invest}$_{N-h}$, as predicted by Theorem 5. Fig. 8 shows the cost $C_A(h)$ borne by the agents as a function of *h*. For the social optimum {all invest}$_{N-h}$, Fig. 8 also shows the total social cost $C_S(h)$ as a function of *h*, for three different assumptions about $C_F(h)$; note that $C_S(h)$ is non-increasing in all three cases. For the sub-equilibrium {none invest}$_{N-h}$,

$C_S(h)$ is decreasing in all three cases; however, the results are not shown in Fig. 8, since for {none invest}$_{N\text{-}h}$, $C_S(h)$ is approximately equal to $C_A(h)$, so would not be clearly visible in the figure.

From Theorems 7 and 9, we know that subsidization of security will in general decrease the total costs $C_A(h)$ borne by the agents in both {all invest}$_{N\text{-}h}$ and {none invest}$_{N\text{-}h}$, and moreover ensures that {all invest}$_{N\text{-}h}$ is the unique sub-equilibrium whenever $h \geq N-k$. Provided that the rate of attacks is sufficiently large that {all invest} would be an equilibrium solution (in the absence of errors), then extensive numerical results suggest that $C_S(h)$ will be non-decreasing in h (as shown in Fig. 8) for both the case in Section 5 and the case of erroneous choice discussed in Section 6. However, we have not been able to prove this. In order to prove this speculation, it would be sufficient to prove that $dC_A(h)/dh \leq -C$, since we know that $C_S(h) = C_F(h) + C_A(h)$. If our speculation is true, this would suggest that in order to minimize the total (expected) social costs, all agents should receive subsidized (free) security (i.e., h=N) in cases where investing in security is the social optimum and security can be provided at lower cost by a third party than by the agents themselves.

8. Conclusions

Sections 2-4 of this paper formulate and solve an interdependent security model for an arbitrary number of agents with attacks occurring over time, focusing on the case of agents with heterogeneous time preferences. Results show that while multiple equilibrium solutions can exist, the social optimum in such cases is for all agents to invest. In order to help achieve this social optimum, the role of tipping and cascading is discussed. In particular, we explore the minimal number of agents who would need to receive subsidized (free) security in order for tipping to occur, and which agents should receive such free security. In this paper, we focus primarily on the effect of discount rates; in particular, the fact that the existence of agents with extreme discount rates can make it undesirable for other agents to invest. However, similar results hold for heterogeneity in other parameters (such as the cost of investing in security, and the loss experienced as the result of a successful attack).

Sections 5-7 further investigate the effect of providing subsidized (free) security on both the stability of equilibrium solutions and the total social cost, in the case of homogeneous discount rates. Results show that subsidization can increase the stability of the socially optimum equilibrium solution (in which all agents invest), reduce or eliminate the adverse effect of erroneous choices, and also decrease the total (expected) social cost of achieving the social optimum. Our work suggests that under appropriate circumstances, providing subsidized security to some agents will: (1) ensure that even agents for which not investing would otherwise be dominant do actually invest (through careful targeting of the subsidies to those agents); (2) lead to tipping and cascading, thereby causing additional agents to invest; (3) increase the stability of the socially optimum equilibrium in which all agents invest; (4) counteract the effect of erroneous choices; and (5) decrease the (expected) total social costs. Thus, it might sometimes be worthwhile for third parties (such as governments) to subsidize the provision of security, or otherwise ensure that the strategy of investing in security is adopted when it is the social optimum, since that might not otherwise occur.

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Table 1: Agent costs at equilibrium

	For investing agents (i=1,…, M), $s_i = 1$		For not-investing agents (i=M+1,…, N), $s_i = 0$	
	$P_i(s_i, s_{-i})$	$P_i(1-s_i, s_{-i})$	$P_i(s_i, s_{-i})$	$P_i(1-s_i, s_{-i})$
Case (1) M=N	C	$L/(1+r_i/\tilde{\lambda})$, $\tilde{\lambda} = \lambda$	N/A	N/A
Case (2) 0<M<N	$C + L/(1+r_i/\tilde{\lambda})$ $\tilde{\lambda} = (N-M)q\lambda$	$L/(1+r_i/\tilde{\lambda})$ $\tilde{\lambda} = \lambda + (N-M)q\lambda$	$L/(1+r_i/\tilde{\lambda})$ $\tilde{\lambda} = \lambda + (N-M-1)q\lambda$	$C + L/(1+r_i/\tilde{\lambda})$ $\tilde{\lambda} = (N-M-1)q\lambda$
Case (3) M=0	N/A	N/A	$L/(1+r_i/\tilde{\lambda})$ $\tilde{\lambda} = \lambda + (N-1)q\lambda$	$C + L/(1+r_i/\tilde{\lambda})$ $\tilde{\lambda} = (N-1)q\lambda$

Table 2: Conditions for an equilibrium in which *M* agents invest

	For investing agents (i=1,…, M), $s_i = 1$	For not-investing agents (i=M+1,…, N), $s_i = 0$
Case (1): M=N	$0 = R_1(0) < r_i < R_2(0)$	N/A
Case (2):* $\max\{N-\tilde{N}, 0\} < M < N$	$R_1(N-M) < r_i < R_2(N-M)$	$r_i < R_1(N-M-1)$ or $r_i > R_2(N-M-1)$
Case (3): M=0	N/A	$r_i \geq 0$

*No equilibrium is possible with $0 < M < \max\{N-\tilde{N}, 0\}$.

Table 3: Conditions for an equilibrium in which *M* agents invest, in terms of the $\{S_l\}$

	$Non(M)$ (Ranges of discount rates of non-investing agents)	$Cmpl(M)$ (Ranges of discount rates not held by any agents)	$Inv(M)$ (Ranges of discount rates of investing agents)
Case (1): M=N	N/A	S_0	$\bigcup_{l=1}^{\min\{\tilde{N}+1, N\}} S_l$
Case (2):* $\max\{N-\tilde{N}, 0\} < M < N$	$\bigcup_{l=0}^{N-M-1} S_l$	S_{N-M}	$\bigcup_{l=N-M+1}^{\min\{\tilde{N}+1, N\}} S_l$
Case (3): M=0	$\bigcup_{l=0}^{\min\{\tilde{N}+1, N-1\}} S_l$	N/A	N/A

*No equilibrium is possible with $0 < M < \max\{N-\tilde{N}, 0\}$.

Figure Captions

Fig. 1: Model structure

Fig. 2: Illustration of the ranges S_l

Fig. 3: Ranges of discount rates possible for *M* agents investing

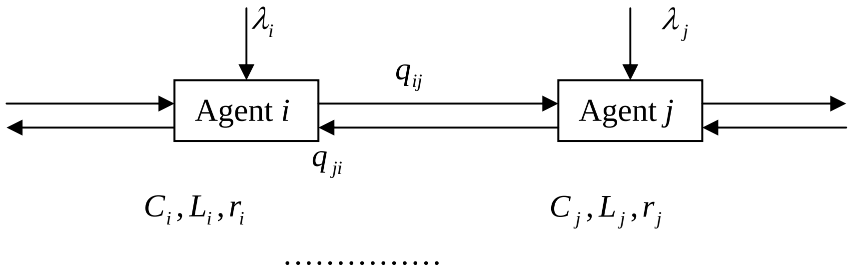
Fig. 4: Ranges of discount rates possible for non-investing agents when *M* agents invest

Fig. 5: Discount rates for which investing and not investing are dominant

Fig. 6: Stability of equilibrium solutions in an *N*-agent homogeneous model

Fig. 7: Stability of the sub-equilibrium solutions $\{\text{all invest}\}_{N-h}$ **and** $\{\text{none invest}\}_{N-h}$

Fig. 8: Total costs of the sub-equilibrium solutions $\{\text{all invest}\}_{N-h}$ **and** $\{\text{none invest}\}_{N-h}$



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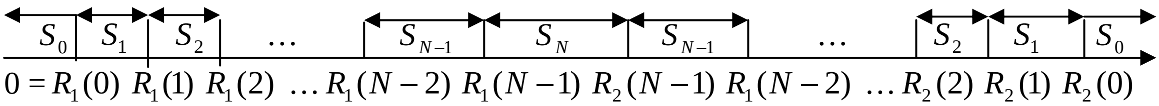
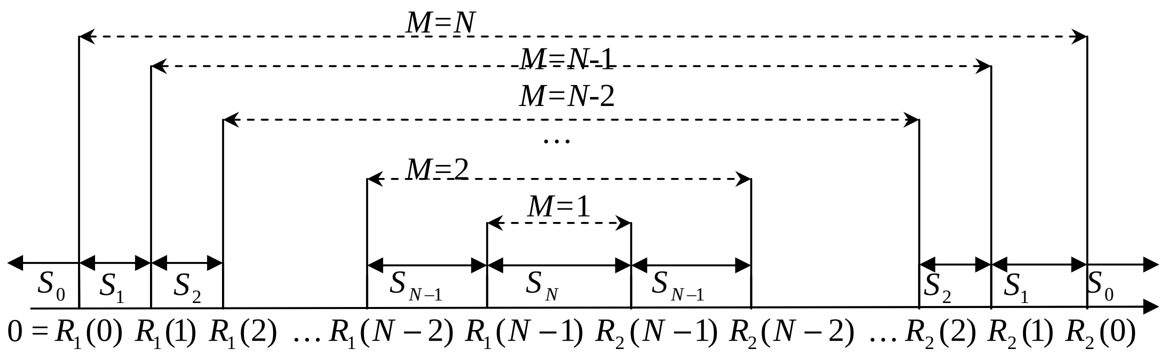
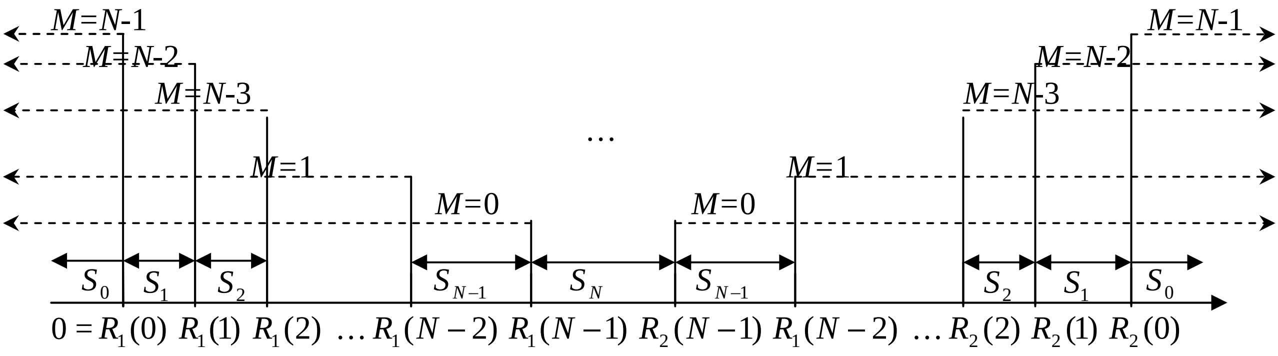
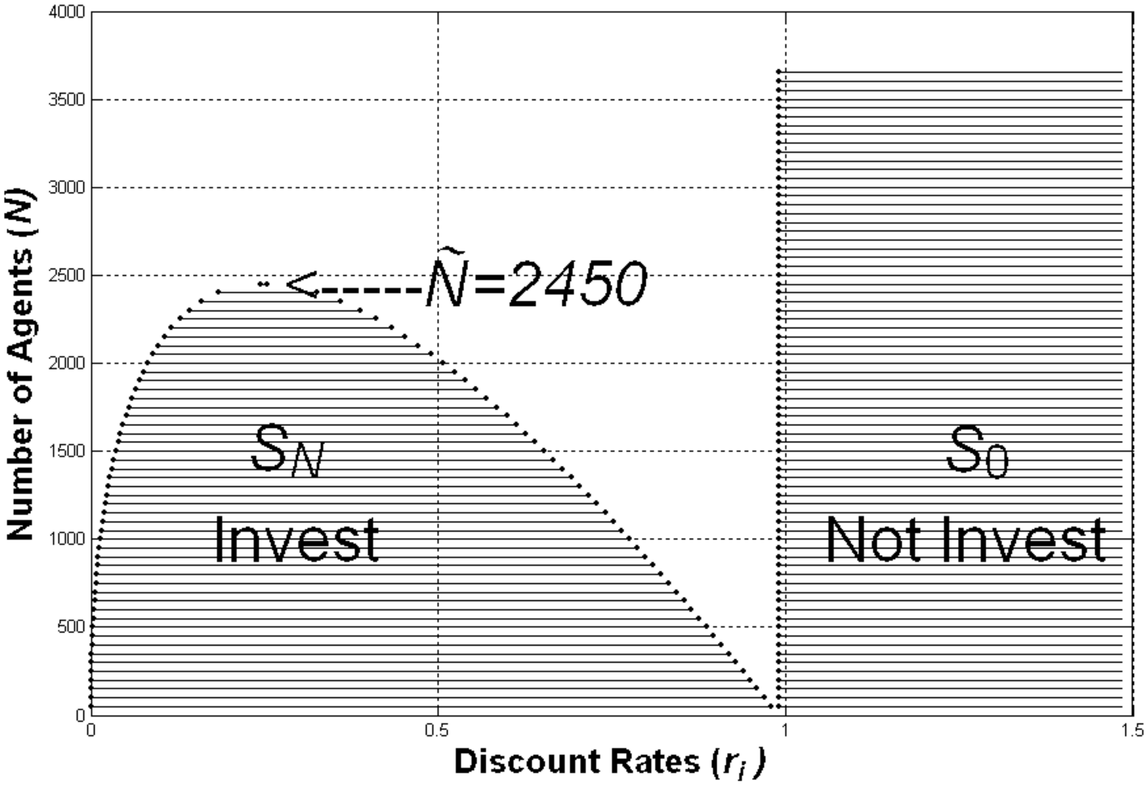
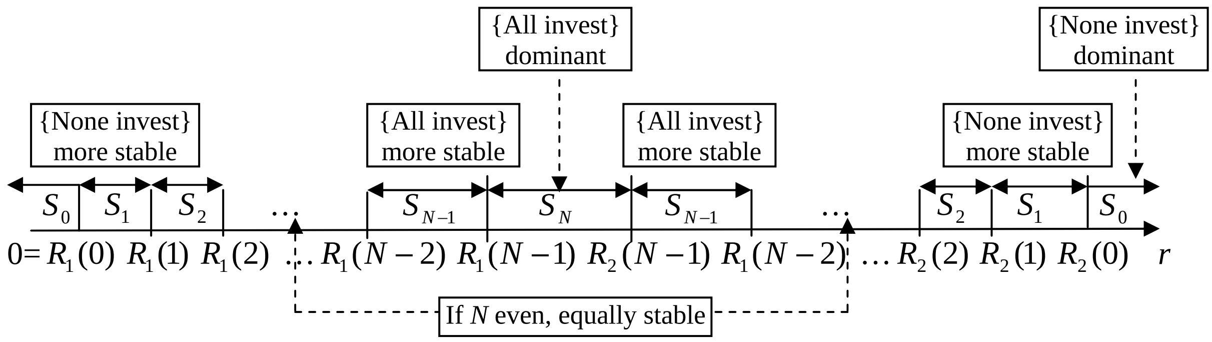

S_0 S_1 S_2 … S_{N-1} S_N S_{N-1} … S_2 S_1 S_0
$0 = R_1(0)$ $R_1(1)$ $R_1(2)$ … $R_1(N-2)$ $R_1(N-1)$ $R_2(N-1)$ $R_1(N-2)$ … $R_2(2)$ $R_2(1)$ $R_2(0)$

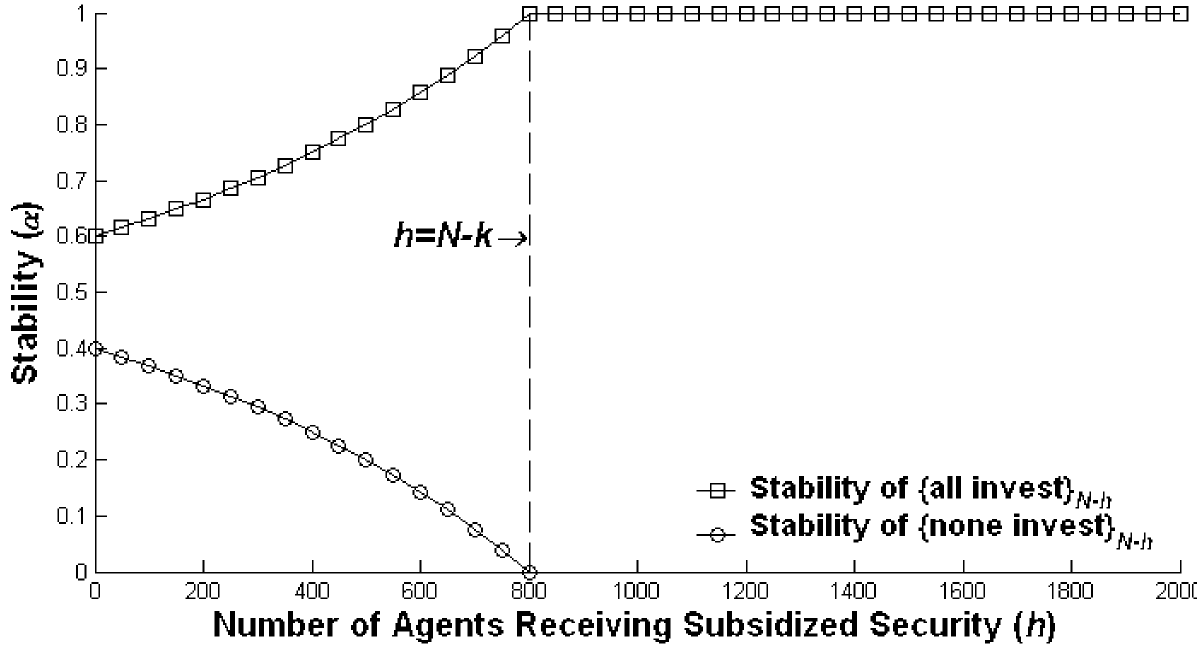

M=N
M=N-1
M=N-2
…
M=2
M=1
S_0 S_1 S_2 S_{N-1} S_N S_{N-1} S_2 S_1 S_0
$0 = R_1(0)$ $R_1(1)$ $R_1(2)$ … $R_1(N-2)$ $R_1(N-1)$ $R_2(N-1)$ $R_2(N-2)$ … $R_2(2)$ $R_2(1)$ $R_2(0)$


M=N-1
M=N-2
M=N-3
M=1
M=0
M=0
M=1
M=N-3
M=N-2
M=N-1
…
S_0 S_1 S_2 S_{N-1} S_N S_{N-1} S_2 S_1 S_0
$0 = R_1(0)$ $R_1(1)$ $R_1(2)$ … $R_1(N-2)$ $R_1(N-1)$ $R_2(N-1)$ $R_1(N-2)$ … $R_2(2)$ $R_2(1)$ $R_2(0)$

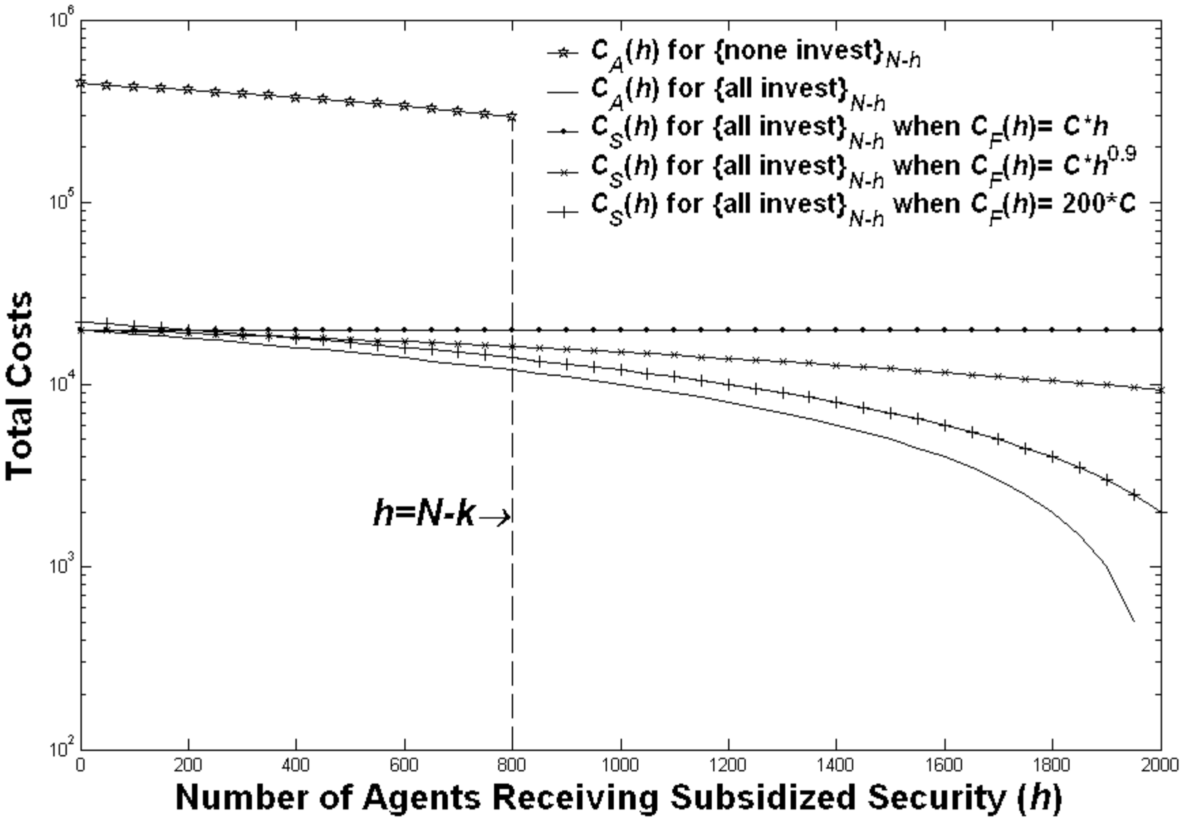


4000
3500
3000
2500
2000
1500
1000
500
0
Number of Agents (N)
0
0.5
1
1.5
Discount Rates (r_i)
<-----$\tilde{N}$=2450
S_N
Invest
S_0
Not Invest


{All invest}
dominant
{None invest}
dominant
{None invest}
more stable
{All invest}
more stable
{All invest}
more stable
{None invest}
more stable
S_0 S_1 S_2 … S_{N-1} S_N S_{N-1} … S_2 S_1 S_0
$0 = R_1(0)$ $R_1(1)$ $R_1(2)$ … $R_1(N-2)$ $R_1(N-1)$ $R_2(N-1)$ $R_1(N-2)$ … $R_2(2)$ $R_2(1)$ $R_2(0)$ r
If N even, equally stable



Stability (α)
Number of Agents Receiving Subsidized Security (h)
h=N-k→
Stability of {all invest}N-h
Stability of {none invest}N-h



$C_A(h)$ for {none invest}$_{N-h}$
$C_A(h)$ for {all invest}$_{N-h}$
$C_S(h)$ for {all invest}$_{N-h}$ when $C_F(h)= C*h$
$C_S(h)$ for {all invest}$_{N-h}$ when $C_F(h)= C*h^{0.9}$
$C_S(h)$ for {all invest}$_{N-h}$ when $C_F(h)= 200*C$
$h=N-k\rightarrow$
Total Costs
Number of Agents Receiving Subsidized Security (h)